PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES
Volume 19 May 15, 1933 Number 5

AKHISSAR SPOTTING OF THE HOUSE MOUSE

By Clyde E. Keeler

Howe Laboratory of the Harvard Medical School and the Bussey Institution

Communicated April 10, 1933

A gray house mouse was trapped* in a grain shop at Akhissar, Turkey, during July of the year 1930. It was a male specimen of Mus musculus and quite ordinary in appearance, save that it bore upon the belly a small white spot. Similar low-grade spottings are frequently encountered in Mus musculus taken in the wild. For example, among the 984 house mice autopsied by Miss Clara Lynch** during the Kern County (Calif.) Mouse Plague (1926–27), 16 bore a large white patch upon the belly, 2 had white tail tips and one possessed both ventral spot and tail markings. Yet although the occurrence of white spotting in the wild is not uncommon, such markings have not been subjected to genetic analysis.

The Turkish specimen was brought to the Bussey Institution where it was mated with a female belonging to an inbred non-agouti, pink-eye, dilute brown strain which carried no spotting factors. This mating produced 7 self mice and 2 having white-tipped tails. These 9 mice were mated *inter se* and gave rise to 8 self, 2 bearing both white-tipped tails and belly spot, and 1 having face-blaze, ventral spot, white feet and white-tipped tail. This heterogeneous F_2 population was allowed to breed at random. They produced many selfs in addition to animals exhibiting all grades of spotting from white-tipped tails to belted piebalds, as shown in figure 1. When high-grade spotted mice appeared they were transferred to other cages where they established pure strains of Akhissar spotting.

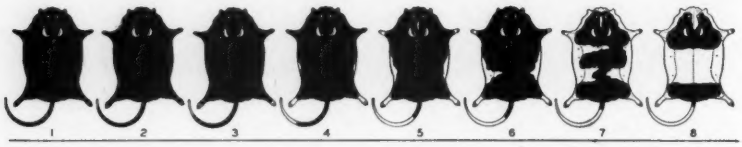
High-grade piebalds of classes 7 and 8 were mated together and produced the population indicated as P_1 in figure 1. They consisted of 2 in class 5, 29 in class 7 and 34 in class 8, but none in lower classes. This suggests that high-grade piebald individuals are purebred for spotting genes, because they throw only high-grade spotted young when mated *inter se.* When high-grade individuals (classes 7 and 8) were crossed to unrelated selfs there were produced F_1 hybrids as follows (see figure 1): 9 self, 15 with

white-tipped tail, 4 with white toes plus white-tipped tail and none of higher classes.

In other F_1 litters not fully inventoried, there occurred two individuals of class 4, having a small belly spot in addition to foot and tail markings. Hence we may conclude that in general the Akhissar gene or genes when hybrid do not produce body spots but are usually limited in their expression to tail tip and feet.

We may infer from the foregoing results of purebreeding and hybridization that wherever this character occurs, purebreds will probably be distributed in classes 5 to 8 and hybrids in classes 1 to 4, inclusive.

When hybrids were mated together an F_2 population was produced covering the entire range from class 1 to class 8 as shown in figure 1. Combining classes 1 to 4 as including self and heterozygous spotted animals, and classes 5 to 8 as probably homozygous spotted, we obtain a ratio of 44 to 16 which approximates a 3:1 ratio and suggests the action of a



P_1					2		29	34
F_1	9	15	4					
F_2	21	7	15	1	2	1	7	6
BC	6	16	8	7	10	4	10	9

FIGURE 1

single recessive gene in the production of Akhissar spotting. Upon this hypothesis we expect an approximation of 45 selfs and heterozygotes to 15 homozygous spotted with which the observed numbers are in close agreement. In an F_2 population produced for linkage study (figure 2) we find 81 selfs or heterozygotes and 34 spotted where the theoretical expectation is 86.25 to 28.75. The validity of this interpretation was tested further by a back-cross of Akhissar spotted individuals (classes 7 and 8) to hybrids. In this experiment there were obtained 37 animals of grades 1 to 4 and 33 of grades 5 to 8, where 35 of each class are expected upon the basis of a single pair of recessive genes. Hence from the close approximation of our results to the expected ratios we conclude that Akhissar spotting is produced by a single recessive mendelizing gene.

Curiously enough, throughout the complete range of its variation Akhissar spotting expresses itself in a fashion quite similar to recessive piebald of the mouse fancy. Like recessive piebald it produces a high

proportion of white faces and belts, with quite regular lines of demarkation between the white and pigmented areas. Breeding results with recessive piebald are similar to those we have obtained for Akhissar spotting by inbreeding and by hybridization in the F_1, F_2 and back-cross generations. These resemblances in expression and breeding behavior suggest that the Akhissar gene may be identical with the recessive piebald gene.

But mere somatic resemblance does not prove identity of determining genes. The short hair characters of the rabbit (Gillet rex, Kurzhaar and Norman rex), for example, have indistinguishable somatic expressions but are due to three different recessive genes. Likewise leaden and blue dilutions of the mouse may only be distinguished by their breeding behavior. In these examples, proof of identity or independence is readily determined by crossing together animals of the two types in question. If the F_1 animals are of the same phenotype as the parents, allelomorphism or iden-

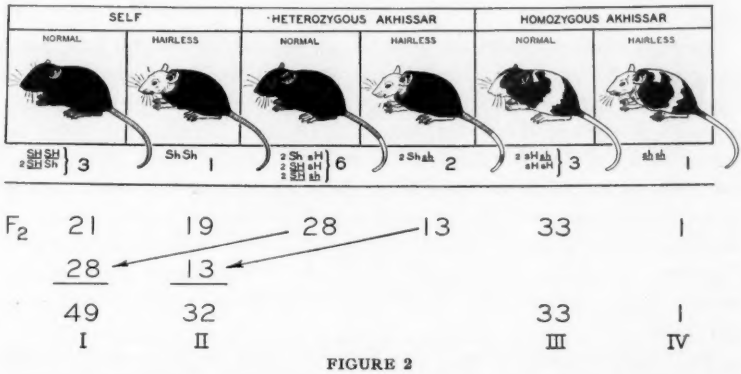


FIGURE 2

tity of the responsible genes becomes probable; if normals are produced, independence is shown.

Not so with the variably expressed spotting genes of mammals, which, even when independent in transmission, usually supplement each other's effects. At present there are under observation in the Bussey Institution mouse colony four different types of spotting all of which may have identical expressions of the type shown in figure 1, class 4; that is, having a ventral spot in addition to white-tipped tail and white hind feet. The crossing together of two such spottings would only result in confusion.

There is, however, a method of determining whether or not a given spotting is identical with recessive piebald. This is by testing its linkage relations with hairless, already employed to prove the genetic independence of dominant (black-eyed white) spotting[1] from recessive piebald. Recessive piebald and hairless are linked. If Akhissar spotting should show

independence from hairless, then the gene for Akhissar and the gene for recessive piebald must lie in different chromosomes. If they show linkage, the two spotting determiners must be located in the same chromosome. In case of linkage, if the cross-over values between Akhissar spotting and hairless are similar to those already found to exist between recessive piebald and hairless, then the two spotting variations are most probably due to genes at the same genetic locus.

A cross was made between Akhissar belted females (Fig. 1, class 8) and unspotted hairless males. The F_1 hybrids were normal haired selfs or white-tipped as expected (grades 1–3, Fig. 1). These were mated *inter se* to provide an F_2 for linkage study. The F_2 population is classified in figure 2, in which for simplicity three groups are recognized as regards spotting, viz., (1) self (unspotted), (2) heterozygous Akhissar and (3) homozygous Akhissar. In each of these three groups occurred both normal and hairless individuals in the numbers indicated below each of the six types.

Allowing H to represent the gene for normal hair, h that for hairless, S the gene for self and s that for Akhissar spotting, we have given in figure 2 the formulae for the genetic types expected and their expected relative frequencies in each of the 6 distinguishable classes. Crossover gametes are underlined. We have indicated the relative sizes of these classes upon the basis of complete independence. The F_2 consisted, as indicated in figure 2, of 21 normal self, 19 hairless self, 28 normal heterozygotes, 13 hairless heterozygotes, 33 normal homozygotes and 1 hairless homozygote. There is obviously a wide departure of some of the classes from their expected relative frequencies. Class 2 is too large and classes 3 and 6 are too small.

To classify this population for an F_2 linkage test, we must combine class 3 with class 1, and class 4 with class 2, and allow classes 5 and 6 to remain as they are. The resultant classes I, II, III, IV should approximate the 9:3:3:1 ratio in case of independence. We actually find 49:32:33:1 where we expect 64.8:21.6:21.6:7.2 upon the basis of free assortment of genes at gametogenesis and random recombination at fertilization. In the event of linkage, class I should be reduced in size because only 2 of the 9 possible genetic combinations composing this class do not contain crossover gametes. Class IV will be reduced because it contains only crossover gametes. The distortion of these ratios is in the direction indicating linkage. The deviation which we have observed, when referred to Immer's Tables,[2] is found to indicate linkage with a crossover value of 14.8% having a probable error of 6.1%.

The work of Snell[3] showed recessive piebald and hairless to be linked with crossover values ranging from 2.6 to 13.3% in his several tests. The difference between these values and mine may be due to errors of random

sampling, differences in genetic constitution of the stocks examined in these two studies, or both.

We conclude that the Akhissar gene and the recessive piebald gene probably reside at one and the same locus. Because the range of expression is the same for these two characters and their manifestation in corresponding crosses is alike, we conclude that they are identical allelomorphs of the same gene.

Summary.—A wild gray house mouse (Mus musculus) bearing a small ventral spot was captured at Akhissar, Turkey. A genetic analysis has shown this spot to be due to the gene which produces the recessive piebald variation so commonly found in laboratory mice.

* This mouse was collected by the author under an appointment from Harvard University as Sheldon Traveling Fellow in Zoölogy.

** We wish to thank Miss Lynch for permission to cite her unpublished data.

[1] Keeler, C. E., *Proc. Nat. Acad. Sci.*, **17**, 101–102 (1931).

[2] Immer, F. R., *Genetics*, **15**, 81–98 (1930).

[3] Snell, G. D., *Ibid.*, **16**, 42–74 (1931).

A MIACID FROM THE SESPE UPPER EOCENE, CALIFORNIA

BY CHESTER STOCK

BALCH GRADUATE SCHOOL OF THE GEOLOGICAL SCIENCES, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated March 18, 1933

In addition to three distinct types of hyænodonts now known from the Sespe Upper Eocene[1] (Locality 150 Calif. Inst. Tech. Vert. Pale.), there remains to be recorded another creodont, namely, a representative of the family Miacidæ. At Locality 150 skull and jaw materials of this form occur nearly as abundantly as do those of the Hyænodontidæ. Special interest attaches to this miacid in view of its relationship to *Pleurocyon*, a genus described by Peterson[2] from the Upper Uinta (Uinta C).

Pleurocyon (Simidectes) merriami, n. subgen. and n. sp.

Type Specimen.—Left ramus of mandible with C, $P\overline{3}$ – $M\overline{3}$, No. 1139 C.I.T. Vert. Pale., plate 1, figures 1, 1*a*.

Paratype.—A maxillary fragment with C, $P\underline{4}$ – $M\underline{2}$, No. 1212, plate 1, figures 2, 2*a*.

Referred Specimen.—Right ramus of mandible with canine but without cheek-teeth, No. 934, plate 1, figures 3, 3*a*, 3*b*.

Locality.—Sespe Upper Eocene, north of Simi Valley, Ventura County, California; Locality 150 C.I.T. Vert. Pale.

Subgeneric and Specific Characters.—Larger than *Pleurocyon magnus* and with dentition decidedly more robust than in the Uinta species. Shorter diastema between C and $P\bar{1}$ than in *P. magnus*. Heel of $P\bar{4}$ without inner ledge. Upper molars with single internal cusp. This species is named for Dr. John C. Merriam in appreciation of his many contributions to the vertebrate paleontology of the Far West.

Description of Material.—The fundamental resemblances between the type from the Sespe and that described by Peterson from the Uinta are seen in (1) one-rooted $P\bar{1}$, (2) two-rooted $P\bar{2}$, but with roots situated close together or fused, (3) $P\bar{3}$ with anterior principal cusp and posterior basal cusp, (4) $P\bar{4}$ with large principal cusp and with posterior basal cusp which is elongated anteroposteriorly and (5) $M\bar{1}$ and $M\bar{2}$ with trenchant heels but with cuspule present on inner border of talonid. The positions of the mental foramina are similar in the two types for the anterior foramen is situated below the posterior end of $P\bar{2}$ and the posterior below the posterior end of $P\bar{3}$.

The crowns of $P\bar{3}$ and $P\bar{4}$ are tilted backward slightly in *P.* (*Simidectes*) *merriami*. A paratype $P\bar{4}$ of *P. magnus* figured by Peterson[3] shows also an inner ledge along the side of the heel, in which respect this specimen agrees with the corresponding tooth of the type of this species. At the anterior base of the principal cusp in this specimen is a style or cusp which is better developed than in the corresponding tooth of *P.* (*Simidectes*) *merriami*. It is separated from the external face of the base of the principal cusp by a vertically directed furrow.

Our type differs from *Oödectes* and *Vulpavus* not only in decidedly larger size, but also in the considerable reduction of $P\bar{2}$ and in the characters of the crowns of $P\bar{3}$ and $P\bar{4}$. The heels of $M\bar{1}$ and $M\bar{2}$ are essentially trenchant, in which respect *Pleurocyon* resembles *Oödectes*, as noted already by Peterson in the comparison of the Uinta material. Although a conule is present at the forward end of the inner side of the heel, the talonid is not basined as in *Vulpavus*. The trigonid region in these teeth is not so high as in *Oödectes* and resembles more that in *Vulpavus*.

No less than seven rami are now known from Locality 150. These specimens exhibit considerable variation in size. Possibly more than one species is represented by this material. The resemblances in the dentition or in the ramus between individual specimens makes it un-

---

DESCRIPTION OF PLATE 1

Pleurocyon (*Sespecyon*) *merriami*, n. subgen. and n. sp.

Figures 1 and 1*a*, type specimen, No. 1139, left ramus, lateral and superior views; figures 2 and 2*a*, paratype, No. 1212, maxillary fragment with superior dentition, lateral and inferior views; figures 3, 3*a* and 3*b*, referred specimen, No. 934, right ramus, lateral, superior and posterior views. All figures $\times 3/4$.

California Institute of Technology Collections. Sespe Upper Eocene, California.

PLATE 1. (Description on opposite page.)

desirable, at least for the present, to recognize more than one type. The referred specimen, No. 934, plate 1, figures 3, 3*a* and 3*b*, belongs to a distinctly larger and probably considerably older individual than the type.

The cheek-teeth have dropped from the alveoli and only the canine remains of the lower dentition. Some reduction in the root regions of $P\bar{3}$ and $P\bar{4}$ has occurred in No. 934 for only the socket for the anterior fang of $P\bar{3}$ and that for the posterior fang of $P\bar{4}$ remain. A low ridge extends between these two alveoli.

The symphysis extends backward to a point below the anterior root of $P\bar{3}$. The jaw displays broad development of the coronoid process and the forward end of the masseteric fossa is deep and well defined. The anterior base of the process is likewise broad laterally. At the upper posterior end the process curves outward. From the outer end of the condyle the posterior border of the jaw extends downward and noticeably inward. The inward deflection of the angle is clearly shown in plate 1, figure 3*b*. In lateral view the angle is seen to be formed by a relatively broad plate of bone which reaches behind the level of the condyle.

The lower canine in both the type and referred specimen possesses a relatively slender crown.

Several skull fragments are likewise available from Locality 150. The upper dentition as characterized by the structural details displayed in individual cheek-teeth of these specimens is clearly the complement of that shown in the lower jaws of *P.* (*Simidectes*) *merriami*. In no instance, however, have we found an upper and a lower jaw in direct association.

In the paratype, No. 1212, the canine and $P\underline{3} - M\underline{2}$ are present. The canine possesses a slender crown, somewhat compressed transversely. Alveoli for $P\underline{1}$ and $P\underline{2}$ indicate that these teeth were single-rooted and the crowns were doubtless considerably reduced. As a matter of fact, additional specimens referred to *P.* (*Simidectes*) *merriami* definitely indicate that this was the case. $P\underline{3}$ is three-rooted with the principal external cusp situated for the most part anterior to the middle transverse line of the tooth. A posterior basal tubercle is present. In addition, a well defined cingulum extends along the inner side from in front of the base of the principal cusp to the inner side of the posterior basal tubercle. At the point where the cingulum passes over the inner root a rudimentary cuspule is formed.

In $P\underline{4}$ the outer half of the crown comprises, in addition to a well developed principal cusp, an anterior ledge or basal cusp and a posterior basal cusp. The inner lobe of this tooth is likewise strongly developed, curving slightly forward in its inward projection. At its inner end the lobe supports a well defined cusp which is separated from the principal cusp and the posterior basal cusp by a broad valley. This tooth differs from $P\underline{4}$

in *Oödectes* in greater reduction of metaconid and in larger size and more posterior position of the inner cusp.

M1 is broader anteroposteriorly than M2 but the transverse diameter is a trifle shorter than that of the latter tooth. The arrangement of the cusps is essentially the same for both teeth. Both molars in *P.* (*Simidectes*) *merriami* possess a single inner cusp (protocone), in which respect the Simi form resembles *Oödectes* and differs from *Vulpavus*. The intermediate conules are present and are clearly shown in M2. In M1 abrasion has evidently obliterated the forward conule. In the illustrations given by Matthew[4] the anterior molars in *Oödectes proximus* possess a forward conule but no metaconule, while in *Vulpavus profectus* both conules are present. In each molar of the paratype of *P.* (*Simidectes*) *merriami* the outer cusps are situated well to the inner side of the outer margin and an external cingulum is absent. The metacone is more subdued than the paracone. A parastyle is present and possesses a slightly more external position with reference to the paracone in M2 than in M1. A third molar of smaller size than M2 is present in the Simi species, but the tooth is not preserved in specimen No. 1212.

The skull fragment represented in No. 1212 indicates that the orbit was evidently of small size. The antorbital foramen, on the other hand, is of relatively large size. The postero-external rim of this opening is situated above the posterior end of M1. The position of this opening is not so far forward as in *Vulpavus profectus*. The expanse of bone lying between the external border of the antorbital foramen and the lower anterior border of the orbit is much less extensive in *P.* (*Simidectes*) *merriami* than in V. *profectus*.

A single upper molar of the paratype of *Pleurocyon magnus* from the Uinta is not completely preserved. This tooth differs from the first or second molars of the Simi species in smaller transverse diameter and in the division of the inner crescent into two cusps of subequal size. Moreover, an external cingulum is well marked in the specimen from the Uinta but is absent in the upper molars of *S. merriami*.

Relationships.—The resemblances between the miacid from the Simi and *Pleurocyon magnus* in lower jaw and lower dentition emphasize their close relationship. The fourth lower premolar and upper molar in the Uinta species display the more important structural characters in which this type differs from the Simi form. *Simidectes merriami* is noticeably larger than the type from the Uinta.

In stratigraphic position, large size, and in the structural characters of the dentition, *S. merriami* represents the most advanced stage of development known to occur within the so-called cercoleptoid division of the Miacidæ. As may be expected, its relationship with the Bridger representatives of this group is much closer than with modern *Cercoleptes*.

Notwithstanding the long period of time separating *Cercoleptes* from *Pleurocyon*, the differences in structural characters known to prevail between these genera are of such magnitude as to obviate the possible derivation of the former type from the latter. *Oödectes* and *Vulpavus* are not so advanced as the Simi form, as shown by their smaller size and by less reduction in the anterior premolar teeth. *Oödectes* resembles our form more closely than does *Vulpavus* in the absence of the hypocone in the upper molars and in the trenchant character of the talonids of the lower molars. In *Pleurocyon magnus* and in *P.* (*Simidectes*) *merriami* the height of the trigonid region of the lower molars is distinctly less than that in *Oödectes*, in which respect these later Eocene forms are more like *Vulpavus*. *Oödectes*, rather than *Vulpavus*, resembles more nearly the ancestral type from which *Pleurocyon* may have evolved.

[1] Stock, C., *Proc. Nat. Acad. Sci.*, **18**, 518–523 (1932).
[2] Peterson, O. A., *Ann. Carnegie Mus.*, **12**, 52 (1919).
[3] Peterson, O. A., *Ibid.*, plate 35, figures 12, 13 (1919).
[4] Matthew, W. D., *Mem. Amer. Mus. Nat. Hist.*, **9**, 344 (1909).

EFFECT OF METHYLTHIONINE CHLORIDE ON PHYTOTOXIC REACTION OF NORMAL AND PATHOLOGICAL BLOOD

BY DAVID I. MACHT

PHARMACOLOGICAL RESEARCH LABORATORY, HYNSON, WESTCOTT & DUNNING, INC., BALTIMORE, MARYLAND

Communicated April 14, 1933

Introduction.—The present investigation was prompted by the brief publications of Matilda Brooks[1,2,3,4] and P. J. Hanzlik.[5] These writers found experimentally that methylthionine chloride, or methylene blue, counteracted the toxic effects of cyanide in animals; and they suggested the use of this dye in cyanide poisoning. Matilda Brooks also suggested the use of methylene blue in monoxide poisoning; and J. C. Geiger[6,7] actually employed this method successfully in cases of both cyanide and carbon monoxide poisoning.

For the past ten years the present writer has been engaged in a study of the effects of normal and pathological blood and of various drugs and toxins on living plant protoplasm. These studies were made particularly on the growth of seedlings of *Lupinus albus* in plant-physiological solutions, with and without blood or other toxic substances studied.[8] In this way it was found that a one per cent solution of normal blood serum from human beings produced an inhibition of from 25 to 30 per cent in the

root growth of *Lupinus albus* seedlings, as compared with the growth of similar seedlings in control physiological solutions. On the other hand, whole blood kept fluid with heparin produced an inhibition of from 30 to 35 per cent. By means of such phytopharmacological tests it was demonstrated that there is a toxin in the blood of women at time of menstruation.[9] Still further studies revealed another toxin present in the blood of pernicious anemia and absent from that of other forms of anemia.[10,11] The average toxicity of a one per cent solution of menstrual blood is 52 per cent and the average toxicity of a one per cent solution of pernicious anemia blood for seedlings of *Lupinus albus* is 45 per cent. These toxins can be differentiated by ultra-violet irradiations[12] and by other tests. Later investigations, carried on by Macht and Pels, who examined a great number of blood specimens from dermatological cases led to the discovery of a toxic reaction in the blood of patients afflicted with the baffling disease known as pemphigus.[13,14] The average phytotoxic index in this case is 54 per cent. In still another paper, Macht showed that the blood of leprosy patients also gives a specific phytotoxic reaction.[15] Finally, in several short communications Macht and his collaborators described a toxic reaction produced by blood in cases of carbon monoxide poisoning and poisoning with illuminating gas.[16,17] In consideration of these findings, it was deemed worthwhile to inquire whether methylene blue produces a detoxification of blood as revealed by the phytotoxic test in cases of carbon monoxide and gas poisoning, on the one hand, and in some of the diseases mentioned above, on the other.

Studies on Carbon Monoxide.—Fresh blood was obtained from the carotid artery of cats under ether anesthesia; and, in order to keep it fluid, it was mixed with a highly purified heparin prepared in these laboratories. This blood was divided into three portions: one was used as a control; a second fraction was saturated with carbon monoxide obtained by treating pure formic acid with sulphuric acid; and a third was saturated with illuminating gas from a gas jet. A one per cent solution of each blood specimen was prepared in a mixture of equal parts of distilled water and Shive's plant-physiological saline,[18] and the toxicity thereof was carefully noted on the growth of single, straight, well-defined roots of young *Lupinus albus* seedlings having an initial length of from 30 to 45 mm. The method of procedure is fully described elsewhere. Briefly, it is as follows: Seeds of *Lupinus albus* are soaked in tap water for twenty-four hours. They are then germinated in finely ground, moist sphagnum moss. When the seedlings have attained a root length of from 30 to 45 mm., they are carefully selected, "matched" and placed in upright test tubes of hard glasss containing control solutions, on the one hand, and unknown substances dissolved in similar solutions, on the other. Ten seedlings are usually employed for each control and other solution examined. The length of

the roots is carefully measured with a millimeter rule, after which they are placed in an incubator and kept at a temperature of from 20° to 21°C. for twenty-four hours. At the end of that time, the length of the roots is again measured and the average increment in growth of the seedlings in both the control and other solutions is determined. The increment in growth of the seedlings in the control solution being regarded as 100 per cent, the indices or coefficients of growth of seedlings in other solutions are expressed as percentages of this figure.

Experiments were made to determine the effect of a one per cent solution of normal whole blood in normal plant-physiological solution. At the same time, experiments were made to determine the effect of small quantities of methylthionine chloride, or methylene blue (1:50,000 to 1:100,000) added to the plant-physiological solution. It was found that the normal toxicity of whole blood is considerably decreased even by weak solutions of the dye. Similar experiments were made with solutions of carbon monoxide blood in normal plant-physiological solution and in a saline containing methylene blue, respectively. Finally, similar experiments were made with whole blood through which illuminating gas had been passed. The results obtained are exhibited in the subjoined table.

TABLE 1

EFFECT OF METHYLENE BLUE ON GROWTH OF LUPINUS ALBUS SEEDLINGS IN NORMAL AND CARBON MONOXIDE BLOOD

KIND OF BLOOD	NUMBER OF EXPERIMENTS	PHYTOTOXIC INDEX OF GROWTH IN NORMAL SHIVE	PHYTOTOXIC INDEX OF GROWTH IN NORMAL SHIVE WITH METHYLENE BLUE, 1:50,000
Normal whole blood (with heparin)	13	65%	81%
CO whole blood (with heparin)	12	45%	72%
"Gassed" whole blood (with heparin)	10	39%	57%

It will be noted that normal whole blood kept fluid with heparin gives an average index of growth of 65 per cent. The same normal blood dissolved in physiological solution with methylthionine chloride, 1:50,000, gives an average index of growth of 81 per cent. Again, the average phytotoxic index given by one per cent solution of blood saturated with carbon monoxide in normal physiological saline is 45 per cent, whereas the same carbon monoxide blood dissolved in saline containing methylthionine chloride, 1:50,000, gives a phytotoxic index of 72 per cent. It may be well to emphasize the fact that in every experiment the tests with normal and pathological blood, with and without methylene blue, were performed simultaneously on seedlings from the same crop and under exactly the same conditions of light, temperature, etc.

Blood treated with illuminating gas, which, of course, contains other gases in addition to carbon monoxide, is always more toxic for plants than pure carbon monoxide blood. The average phytotoxic index of growth of

seedlings in normal saline solution was 39 per cent. When such blood was dissolved in physiological saline containing methylthionine chloride (1:50,000), however, the average phytotoxic index was 57 per cent. It is quite evident that the addition of methylthionine chloride to the solutions of "gassed" blood results in a marked detoxification of carbon monoxide blood; and the same is true of blood treated with illuminating gas. These data agree completely with the results obtained in the animal experiments of Matilda Brooks and J. C. Geiger and offer an additional support to her suggestion regarding treatment of monoxide poisoning in human beings.

Experiments on Pathological Blood Sera.—It is not necessary to employ whole blood from patients suffering with such diseases in order to determine their characteristic toxicity. Blood serum has been used almost exclusively by Macht and his collaborators in phytopharmacological studies of these diseases. Experiments were made with samples of pemphigus serum, one per cent, dissolved in normal physiological saline and in physiological saline to which methylene blue had been added. Similar experiments were made with normal blood serum of human beings and with blood sera from cases of leprosy, pernicious anemia and menotoxin. The results obtained are exhibited in table 2. It will be noted that

TABLE 2

EFFECT OF METHYLENE BLUE ON GROWTH OF LUPINUS ALBUS SEEDLINGS IN NORMAL AND PATHOLOGICAL SERA

KIND OF BLOOD SERUM	NUMBER OF EXPERIMENTS	PHYTOTOXIC INDEX OF GROWTH IN NORMAL SHIVE	PHYTOTOXIC INDEX OF GROWTH IN NORMAL SHIVE WITH METHYLENE BLUE, 1:50,000
Normal	15	73%	89%
Pemphigus	15	55%	69%
Menotoxin	2	52%	52%
Leprosy	10	43%	65%
Pernicious Anemia	10	45%	73%

methylene blue produces a distinct detoxification of normal blood serum, as indicated by the phytotoxic test on *Lupinus albus* seedlings. It will be further noted that when treated with methylthionine chloride, 1:50,000, blood sera from cases of pemphigus were rendered less toxic by about 14 per cent. A marked detoxification of leprosy serum was likewise produced by such solutions of methylthionine chloride. The average of ten sets of experiments gave the toxicity of a one per cent solution of leprosy serum in normal physiological saline as 43 per cent, while the same serum with the addition of methylene blue gave a phytotoxic index of 65 per cent. In case of pernicious anemia sera, the average of ten sets of experiments gave 45 per cent as the phytotoxic index of seedlings grown in one per cent pernicious anemia serum in normal Shive's solution, while seedlings grown in the same sera at the same time in Shive's solution with methylene blue

added thereto gave a phytotoxic index of 73 per cent. Curiously enough, the serum from menstrual blood was not rendered less toxic by methylene blue in the few experiments performed by the author. This finding was not altogether unexpected; in fact, it was predicted by the writer before the test was made that menotoxin would probably not be affected to any extent by methylene blue because all the evidence so far accumulated in regard to the chemical nature of menotoxin points to its probable relation to oxycholesterin.

The results obtained in this preliminary study of toxic blood sera, and especially of those from cases of pemphigus, pernicious anemia and leprosy, are not only of scientific interest but may possibly be of considerable practical importance. At the present stage of experimental therapeutics the physician has several efficient agents wherewith to combat the diseases of leprosy and pernicious anemia. In the former, chaulmoogra oil and chaulmoogra esters have proved efficacious while in the latter, liver substance and liver extracts, on the one hand, and irradiation with certain ultra-violet rays,[19] on the other, have given a measure of relief. In case of pemphigus, however, the dermatologist is completely at a loss and much in need of some rational method of procedure.

One cannot help, therefore, entertaining the idea of a possible indication for the use of methylene blue in such clinical cases. The toxicology of this dye has been described by a number of the older writers, such as Meyer,[20] Caseneuve,[21] Ehrlich and Lippmann[22] and Garfounkel and Gautrelet.[23] However, an examination of the more complete bibliography on the toxicity of methylene blue, as presented by Fühner,[24] and by Flüry and Zernik,[25] reveals that the toxic and lethal dosages given by various investigators differ extensively. Working with chemically pure methylthionine chloride (medicinal, U. S. P.), the present writer found the minimal lethal dosage on intravenous injections for rabbits to be from 20 to 25 mg. per kg., given in one per cent solution; and the lethal dosage on intravenous injection for cats under ether to be 40 mg. per kg. Geiger employed in his experiments a solution of methylene blue in distilled water. The dosage employed in these clinical cases was probably too high. In the writer's opinion, it would be advisable to begin clinical administration of methylene blue solutions by vein with much smaller doses of the drug, preferably in isotonic solution. In view of the experiments described above, it is reasonable to suppose that even very small doses of the drug may be pharmacologically effective. Experiments performed by the writer on the toxicology of methylene blue, which will be published elsewhere, revealed that different samples of the drug obtained on the market varied greatly in their potency and toxicity; and it is quite evident that if this drug is to be introduced for intravenous administration in therapeutics, a standard method of assaying it should be established.

Summary.—1. The effects of weak solutions of methylthionine chloride, or methylene blue, in normal plant-physiological saline and in physiological saline containing whole blood or blood serum were studied on the growth of *Lupinus albus* seedlings.

2. It was found that solutions of methylthionine chloride, 1:50,000 to 1:100,000, decreased the phytotoxic properties of normal whole blood and of normal blood serum.

3. It was also found that such solutions of methylene blue detoxify carbon monoxide blood, as measured by its phytotoxic reaction, thus confirming the results obtained on animals by other investigators.

4. It was found that methylene blue, added to solutions of sera from pernicious anemia, leprosy and pemphigus, decreased the toxicity of these sera as studied by the phytopharmacological method.

5. The laboratory experiments described above raise the question of the possible usefulness of methylene blue as a therapeutic agent in some obscure diseases characterized by a toxic reaction of the blood, as well as in carbon monoxide poisoning. The investigator in this field is cautioned, however, to bear in mind that methylene blue is not an innocuous drug and to begin all clinical experiments made in this connection with very small doses of the dye.

[1] Brooks, *Proc. Soc. Exper. Biol. and Med.*, **29,** 1228(1932).
[2] Brooks, *Amer. Jour. Physiol.*, **102,** 145 (1932).
[3] Brooks, *Proc. Soc. Exper. Biol. and Med.*, **30,** 493 (1933).
[4] Brooks, *Amer. Jour. Physiol.*, **104,** 139 (1933).
[5] Hanzlik, *Jour. Amer. Med. Assoc.*, **100,** 357 (1933).
[6] Geiger, *Ibid.*, **99,** 1944 (1932).
[7] Geiger, *Ibid.*, **100,** 1103 (1933).
[8] Macht, *Science*, **71,** 302 (1930).
[9] Macht and Lubin, *Jour. Pharmacol. and Exper. Therap.*, **22,** 413 (1924).
[10] Macht, *Ibid.*, **29,** 461 (1926).
[11] Macht, *Festschrift of Professor Emil Bürgi*, Basel, 228 (1932).
[12] Macht, *Proc. Soc. Exper. Biol. and Med.*, **24,** 966 (1927).
[13] Macht and Pels, *Arch. Dermatol. and Syphilol.*, **19,** 640 (1929).
[14] Macht and Pels, *Ibid.*, **23,** 601 (1931).
[15] Macht, *Acta Dermatologica*, **18,** 126 (1932).
[16] Macht, Blackman and Swigart, *Proc. Soc. Exper. Biol. and Med.*, **21,** 287 (1924).
[17] Macht, *Science*, **66,** 198 (1927).
[18] Shive, *Physiological Researches*, **1,** 327 (1915).
[19] Macht and Anderson, *Jour. Pharmacol. and Exper. Therap.*, **34,** 365 (1928).
[20] Meyer, *Arch. mikro. Anat.*, **46,** 282 (1895).
[21] Caseneuve, *La coloration des vins*, Paris, 87 (1886).
[22] Ehrlich and Lippmann, *Deutsche med. Wchnschr.*, **16,** 493 (1890).
[23] Garfounkel and Gautrelet, *Compt. rend. l'Acad. sc.*, **157,** 1464 (1913).
[24] Fühner, *Handbuch der experimentellen Pharmakologie*, A. Heffter, Berlin, I, 1268 (1923).
[25] Flüry and Zernik, *Handbuch der biologischen Arbeitsmethoden*, Emil Abderhalden, Abt. IV, Teil 7, Heft 7, Lieferung 251, 1366 (1928).

A PHYSICAL ANALYSIS OF WOLF-RAYET SPECTRA

By Cecilia H. Payne

Harvard College Observatory, Cambridge, Massachusetts

Communicated April 15, 1933

The Wolf-Rayet stars have long been known to occupy the head of the spectral sequence, partly because of their close affiliation with the absorption O stars,[1] partly because they are the most extreme of the bright-line stars.[2] Their position in the spectral sequence carries the implication of very high temperature, though they are evidently inaccessible, from their spectral peculiarities, to the ordinary methods of temperature determination.[3]

The chief obstacle to a physical analysis of the Wolf-Rayet stars has been our ignorance of the source of many of the bright lines in their spectra, an obstacle removed during the past year by the progress of the laboratory analysis of the spectra of the stripped lighter atoms, notably in the hands of Edlén.[4] As a result of Edlén's work it has proved to be possible to identify almost all the lines recorded in the spectra of the Wolf-Rayet stars.

The atoms that give lines identified in Wolf-Rayet spectra are as follows: *H, He I, He II, C II, C III, C IV, N II, N III, N IV, N V, O II, O III, O IV, O V, O VI*?, *Al II, Al III, Si II, Si III, Si IV, P III, S II* and *S III*. All the accessible strong lines arising from these atoms have been identified. No "forbidden" lines have been recognized in the Wolf-Rayet spectrum. The identifications are given in a paper that will appear shortly.[6]

Two important observational facts emerge from the analysis: (1) The stars fall unambiguously into two groups—those whose spectra contain carbon, and those whose spectra do not. This observation has been made simultaneously and independently by Beals.[7] (2) The bright-line spectra of many of the stars contain lines arising from three, or even more, successive stages of ionization of the same atom, and there is a range in excitation energy of perhaps a hundred volts for lines simultaneously present in some of the spectra.

The line analysis is a necessary preliminary to an extended study of physical constitution, but three simple conclusions can at once be drawn.

(1) The temperatures of the Wolf-Rayet stars must be high—even higher than their position in the spectral sequence, above the *B* stars, might imply. Application of the methods of evaluating "nebular" luminosity, developed by Zanstra[8] and Beals,[9] to the newly identified atoms lead to minimum temperatures of over 100,000° for the hottest of the Wolf-Rayet stars.

(2) The large range in excitation energy within one spectrum shows

that we are not dealing with ionization in a stellar atmosphere, but essentially with a compact planetary nebula, stratified as the larger planetaries are stratified, with the lines of lowest excitation energy at maximum strength furthest from the exciting star.[10]

(3) There are real differences in abundance of atomic nuclei, some Wolf-Rayet stars containing little or no carbon, others displaying large quantities. The absence of carbon lines is probably not caused by a physical suppression of the spectra of carbon. Normally, at any temperature between 30,000° and 100,000° at least one of the three spectra *C II*, *C III* or *C IV* would be in receipt of the excitation energy appropriate for the production of its lines. As judged from the other atoms represented, the temperatures of the Wolf-Rayet stars are distributed throughout this range. Furthermore, the three principal atoms represented (other than hydrogen and helium) are the three atoms of successive atomic number, carbon, nitrogen and oxygen. These three nuclei provide three isoelectronic sequences (*C II, N III, O IV; C III, N IV, O V; C IV, N V, O VI*) each consisting of three spectra precisely similar in structure, and differing only in energy of excitation. It would be easier to account for the suppression of all the members of an isoelectronic sequence than for the suppression of the spectra of successive ionization stages of one atom. But in many stars one or two members of an isoelectronic sequence are present, while the third (the one corresponding to carbon) is absent. It is difficult to picture any cause for the existence of two groups of spectra other than an actual difference in abundance of atomic species. A more detailed discussion of this point, employing also evidence from other astrophysical sources, will be published elsewhere.

The identification of the lines in the Wolf-Rayet spectra provides a starting point for physical analysis. When the minimum temperatures of the exciting stars are known, it is possible to make a specific picture of the associated "nebula," and to relate it to the more tenuous nebulae that surround the nuclei of planetaries. The interpretation of the line profiles of Wolf-Rayet stars,[11] in terms of continuous outflow of their atmospheres, fits well into this picture, as do the existence of hydrogen envelope stars and of stars that are intermediate between bright-line *B* stars and planetary nebulae.[12] The relation of the Wolf-Rayet stars to planetary and diffuse nebulae, and to novae, is under investigation in collaboration with Dr. D. H. Menzel and Dr. J. C. Boyce. The chief conclusion of the present summary is that the Wolf-Rayet stars represent a special case of the problem of the nebular spectrum.

[1] J. S. Plaskett, *Publ. Dom. Ap. Obs.*, **2**, No. 16 (1924); C. H. Payne, *Harv. Circ. 263* (1924).

[2] Cf. C. H. Payne, *The Stars of High Luminosity*, 101 (1930).

[3] Gerasimovič, *Harv. Circ. 340* (1928).

[4] B. Edlén, *Observatory*, **55**, 115 (1932).

[5] I am greatly indebted to Dr. Bengt Edlén for communicating to me his unpublished term values for *N IV*, *N V*, *O V* and *O VI*.

[6] C. H. Payne, *Zeit. für Astrophys.* (in press).

[7] Private communication.

[8] H. Zanstra, *Publ. Dom. Ap. Obs.*, **4**, No. 15 (1931).

[9] C. S. Beals, *Mon. Not. Roy. Astr. Soc.*, **92**, 677 (1932).

[10] I. S. Bowen, *Astrophys. Journ.*, **67**, 1 (1928).

[11] C. S. Beals, *Publ. Dom. Ap. Obs.*, **4**, No. 17 (1930); D. H. Menzel, *Publ. Astr. Soc. Pac.*, **41**, 344 (1929).

[12] P. W. Merrill, *Publ. Astr. Soc. Pac.*, **44**, 123 (1932).

NOTES ON BLOOD-GROUPS AND RACE IN THE PACIFIC

By William W. Howells

Department of Anthropology, American Museum of Natural History

Communicated April 4, 1933

This paper summarizes some of the results of a study based upon material supplied by Doctor William L. Moss, who as a member of the Crane Pacific Expedition of 1927, collected anthropological measurements and blood types in Fiji, in the Solomon Islands and on the northeast coast of New Guinea. A more complete account of the findings, together with the anthropometric data, will be published later.

When blood-typing first began to be carried out on a world-wide scale, anthropologists took hope that the blood-groups would furnish them with a new and independent approach to racial determination and the study of racial history. This hope, however, has been given little encouragement by subsequent revelations. In the first place, the genetical mechanics of the blood-groups continue to elude definition. It has recently been revealed that there are two separate *A* genes instead of one, which renders still more uncertain the significance of all the material heretofore collected. In the second place, the more the mass of data is enlarged, the more contradictory it becomes. (Steffan and Wellisch, in the *Zeitschrift für Rassenphysiologie*, have tabulated the results of all the work done up to a short while ago.) Different samples collected from people of supposedly the same racial composition show surprising differences in the percentages of the blood-groups obtaining; probably this is due partly to faulty type determinations, and partly to the effects of inbreeding, though the latter is one of the most nebulous aspects of the whole problem. This wide variability renders it impossible to place any extended interpretation upon the percentage distribution of a single group.

Below are given the percentages of the four blood-groups in Doctor

Moss's Melanesian material, together with most of the existing data for the Pacific.

	NO.	GROUP O	GROUP A	GROUP B	GROUP AB
Fiji	160	43.8	43.1	9.4	3.8
Solomon Islands	107	49.5	31.8	16.8	1.9
New Guinea	59	42.4	39.0	13.6	5.1
Hawaii (Nigg)	413	36.5	60.8	2.2	0.05
Samoa (Nigg)	51	41.1	37.5	13.7	7.9
Central and South Australia (Cleland)	355	43.4	56.6	0.0	0.0
Micronesia (Steffan and Wellisch)	2259	47.9	30.4	18.0	3.7

Disregarding Australia, it is apparent in a rough way that proceeding from east to west the amount of *A* decreases and the *B* increases, while the size of the *O* group remains fairly constant. This progression continues in Indonesia and Japan, where the *B* and *A* groups, though varying considerably, are roughly equal in size; an exception to this is made by the Negritos of the interior of the Philippines, where the proportions are more like those of Melanesia. In general, Melanesia is definitely lower in *B* and higher in *A* than the fringe of Asia, while in Hawaii the *B* element is nearly lacking.

Several attempts have been made to relate the blood-groups to such somatic characters as pigmentation, head form, etc. Some authors, taking into consideration large areas, such as Russia, have shown that subgroups with a greater percentage of blond hair or dolichocephaly have also a greater percentage of Group *A*, but this is not direct proof of association, being probably only the manifestation of geographical variation. Petrow, and Semenowa, *et al.*, claim to have obtained satisfactory direct correlations between blood type and both pigment and head form. Their efforts are too involved for discussion here, and must be dismissed with the statement that the writer believes the methods employed to be invalid.

In treating the present material, the Fiji and Solomon Islands series were each divided into their *A*, *B* and *O* groups, and the means and probable errors for all available measurements and indices were calculated for all the groups; these were stature, sitting height, relative sitting height, head length and breadth, cephalic index, face breadth and height, cephalo-facial and facial indices, nasal height, breadth and index. (The *AB* group had to be discarded on account of its small size.)

In both series the differences between the groups in each corresponding mean was obtained and compared to its probable error. This produced no evidence whatever of statistically significant differences between the blood groups. The following table gives the number of times the difference between corresponding means was less than once its probable error, more than once, more than twice, or more than three times; these distributions are to be compared with the normal distribution for thirteen characters, as determined by Goring.

	DISTRIBUTION OF XPE'S				
	0–1	1–2	2–3	3–4	MEAN
Normal	6.50	4.16	1.82	0.43	1.23
Fiji					
A and *B*	8	5	0	0	0.95
A and *O*	11	2	0	0	0.65
B and *O*	8	4	1	0	1.04
Solomons					
A and *B*	8	3	2	0	0.98
A and *O*	5	6	1	1	1.22
B and *O*	9	3	1	0	0.86
Totals					
Normal	19.50	12.48	5.46	1.29	1.23
Fiji	27	11	1	0	0.88
Solomons	22	12	4	1	1.02

It is readily seen that the group differences are in general rather less than those which mere chance would provide. In these two regions, at least, there is no basis for the idea that blood type may be associated with any measurable character.

The principal facts of importance to anthropology furnished by the blood-groups have a broad bearing on relative chronology. By and large, it appears from the distribution of both, that the *A* factor is older than the *B*. It is further apparent that America was peopled before either had originated, since the Indians have a very low percentage of both factors, the amount being least of all in the purest groups (Snyder). In the same way the aborigines of Australia lack the *B* factor, or in some places possess it in very small amounts, while they have a high percentage of *A*, which indicates that the natives, or the last comers at any rate, arrived in the period between the appearances of the *A* and *B* genes.

It is to be noticed that Hawaii, where the *B* factor is very small, has a high percentage of *A;* the same situation prevails in Australia, and in one of the Pelew Islands, Sonsol, from both of which localities the *B* group is entirely absent, and where the *A* group comprises 56 per cent and 67 per cent of the population, respectively. This indicates that the *A* group rises to high proportions when the *B* gene is not present, and is materially cut down when the *B* gene is introduced, the *O* group being little affected in size by this last event. If this is the case, it is probable that the people of the Pacific once had only the *A* element, and that the three groups named above are holdovers from this era, supposing the *B* to have been only recently introduced into Hawaii. Therefore, it is probable that when the Polynesians emigrated from their original homeland, about two thousand years ago, the *B* element had not arisen, or at any rate had not reached them. Nor had it arrived in Melanesia at this date, since the Polynesians probably came from southeast Asia, where they would have acquired the

B element as it passed through. Therefore, though it may have been later carried into Melanesia directly, it is likely that it reached Polynesia by diffusion.

Cleland, J. B., "The Blood-Grouping of Central Australian Aborigines," *Jour. Trop. Med. Hyg.*, **34**, 353–358, London (1931).

Goring, C., *The English Convict. A Statistical Study*, London, 1913.

Nigg, Clara, "A Sudy of the Blood-Group Distribution among Polynesians," *Jour. Immunol.*, **19**, 93–98, Baltimore (1930).

Petrow, G. J., "Zur Frage des korrelativen Zusammenhanges anthropometrischer Merkmale mit den Blutgruppen," *Ukrainisches Zentralblatt Blutgrupperforschungen*, **5**, 73–88, Kharkof (1931).

Semenowa, L. F., Nasajew, and Kalinina, A. W., "Über den Zusammenhang der Pigmentierung, des Kopf und Gesichtsindexes mit den Blutgruppen in der Russischen Bevölkerung," *Ukrainisches Zentralblatt Blutgruppenforschungen*, **5**, 96–106, Kharkof (1931).

Snyder, L. H., "Human Blood-Groups, Their Inheritance and Racial Significance," *Amer. Jour. Phys. Anthrop.*, **9**, 233–263, Washington (1926).

Steffan, P., and Wellisch, S., "Die Geographische Verteilung der Blutgruppen," *Zeit. Rassenphysiol.*, **1**, 46–60, 80–81, 154–159; **2**, 114–115; **3**, 184–187, München (1928–1931).

THE EXCITATION POTENTIAL OF THE λ 2883 AND λ 2895 BANDS OF CARBON DIOXIDE

BY ANGUS S. ROY* AND O. S. DUFFENDACK

DEPARTMENT OF PHYSICS, UNIVERSITY OF MICHIGAN

Communicated March 31, 1933

There have been several investigations on the emission spectra of carbon dioxide, but although some progress has been made on the analysis of the bands, it is still a matter of doubt whether the emitter of this spectrum is the neutral molecule or the carbon dioxide ion, CO_2^+, or if part of the spectrum is due to the one and part to the other.

The purpose of the present investigation is to attempt to excite the CO_2 emission spectrum in mixtures containing an excess of oxygen and a trace of carbon monoxide or carbon in the discharge tube, and to ascertain more definitely the emitter of the two bands λ 2883 and λ 2895 of the carbon dioxide system.

The emission spectrum of carbon dioxide was first studied by Fox, Duffendack and Barker,[1] who observed the spectrum of a low voltage arc in a tube through which carbon dioxide continually flowed. Most of the bands found by them have since been identified as bands of the "second negative group of carbon" as reported by Kayser[2] and later by Hof[3] and Bair.[4] The same system of bands was obtained by Duffendack and

Smith[5] in He-O_2-CO and Ne-O_2-CO mixtures but not in A-O_2-CO mixtures, and from their results they deduced that the so-called carbon dioxide bands were really due to the carbon dioxide ion, CO_2^+. Duncan[6] has examined the two bands λ 2883 and λ 2895 under high dispersion.

More recently Smyth[7] has attempted to analyse the system of bands into a number of progressions. In his latter paper he concluded, taking into account several experiments of his own and some measurements of the absorption spectrum of CO_2 by Henning,[8] that the spectrum was due to the neutral molecule. The experiments of Smyth showed that there existed a close relation between several of the carbon monoxide systems and the carbon dioxide spectrum, small changes in the conditions of excitation causing considerable changes in the relative intensity of the different systems. Smyth assumed that the CO systems were excited directly from the normal CO_2 molecules; that is to say, that the CO_2 molecules are dissociated and the resulting CO molecules excited in a single act. From this he concluded that the excitation potential of the CO_2 bands was 16.6 volts. Such a value would therefore mean that the initial levels of these bands was above the ionization potential of CO_2, which is 14.4 volts,[9] and the final levels would then be below the ionization potential. These bands must, therefore, originate in a negative term more than two volts above the ionization potential of CO_2. There is at present no means of predicting such a term, but it is rather surprising to find one indicated so much as two volts above the ionizing potential.

Henning found in his absorption measurements a series of equally spaced bands at wave-lengths corresponding to 15.7 to 16.7 volts; Smyth now identifies the upper levels of Henning's absorption bands with the initial levels of the emission bands of CO_2. However, as Smyth himself has pointed out, this leads to several difficulties and his original scheme of energies for the bands is subject to considerable correction if these levels are the same.

Schmid[10] and Mulliken[11] have attempted to interpret the rotational structure of some of the bands of carbon dioxide. The bands revealed a simple structure, consisting of P and R branches of equal intensity. Mulliken shows that the bands are most probably of the type $^1\Pi \longrightarrow {}^1\Pi$, but possibly $^1\Sigma \longrightarrow {}^1\Sigma$, or much less likely $^2\Sigma \longrightarrow {}^2\Sigma$. If either of the first two is correct, then the emitter of the bands is the neutral molecule, but if $^2\Sigma \longrightarrow {}^2\Sigma$, then the emitter must be CO_2^+. This definitely agrees with Smyth's statement that the spectrum is due to the neutral molecule. At the end of the paper Mulliken makes a casual suggestion concerning the two bands λ 2883 and λ 2895. He points out that the bands have P, Q and R branches and all the band lines have pronounced Zeeman effects, and so they may be interpreted as the (O—O) band of a $^2\Pi \longrightarrow {}^2\Pi$ transition of CO_2^+.

The two bands λ 2883 and λ 2895 have appeared in lists of other bands. They were placed by Jevons[12] and Johnson[13] along with the first negative bands of CO; also it is very probable that the lines which appear as unclassified lines in Fowler's analysis of OII,[14] are due to these bands. Emeleus[15] has also found these bands in the negative glow in an oxygen discharge, when no other spectra due to carbon compounds were present. More investigations seemed therefore necessary in order to establish experimentally the nature of the emitter of these two bands.

Experimental.—The discharge used to excite the spectrum was a low voltage arc. The carbon monoxide had been generated outside the system and contained a mere trace of nitrogen as impurity. The oxygen was obtained from very pure potassium permanganate.

Two spectrographs were used, a constant deviation quartz Hilger E32, and the quartz Hilger E1.

The tube was operated at a pressure of about 0.3 mm. of Hg. It was extremely difficult to keep the arc current above about 12 milliamperes without flares occurring and so almost all the exposures were made with an arc current about 10 milliamperes, when the plate voltage was between sixty and seventy volts.

Results and Discussion.—At first, mixtures at a total pressure of 0.3 mm. of Hg containing oxygen with only a trace of carbon monoxide were excited. These mixtures contained from 10^{-4} to 10^{-5} mm. of carbon monoxide. The spectrum was photographed in the visible and the ultra-violet regions. From the plates it was found that the carbon dioxide spectrum, as obtained by Fox, Duffendack and Barker, was developed, and, at the same time, several systems of carbon monoxide were excited. The third-positive bands of carbon monoxide were strong; the Ångström bands were also strong and the Comet Tail bands due to CO^+ were present,

TABLE 1

SYSTEM	EMITTER	O_2-CO MIXTURES INTENSITY
Second Negative Bands of Oxygen	O_2^+	strong
Ultra-violet Bands of Oxygen	O_2^+	absent
Third Positive Bands of CO	CO	medium
Ångström Bands	CO	strong
Comet Tail Bands	CO^+	weak
Baldet-Johnson	CO^+	absent
Negative System	CO^+	absent
Carbon Dioxide Bands	CO_2	medium
λ 2883, λ 2895	CO_2^+	medium

but weak. Neither the Baldet-Johnson bands nor the first negative bands of CO^+ were excited. On all the plates, when a trace of carbon monoxide was present in an excess of oxygen, the above systems always appeared. Table 1 gives a list of the band systems which were excited.

These results are in agreement with Smyth's observation and seem to

point to the fact that the excitation potential of most of the carbon dioxide system must be close to 16.5 volts, provided that the carbon monoxide systems are assumed to be excited by dissociating the CO_2 molecule and simultaneously exciting the CO molecule in a single act. The carbon dioxide would be formed in the neighborhood of the filament by the reaction of oxygen with the carbon monoxide. The energy diagram in figure 1 shows the energy levels of carbon dioxide and carbon monoxide calculated from normal CO_2, and shows the very close agreement, provided the above manner of excitation is correct.

Using the same mixtures of carbon monoxide and oxygen, whenever the current density was increased the relative intensity of several systems changed with respect to each other. With increase in current the negative bands of carbon monoxide together with the first negative bands of nitrogen were strongly excited; at the same time the ultra-violet bands of oxygen now appeared and the two bands λ 2883 and λ 2895 of carbon dioxide were enhanced markedly relative to the other carbon dioxide bands and the two neighboring bands λ 2832 and λ 2976 of the third positive system of carbon monoxide. This suggested that the excitation potential of the two carbon dioxide bands, λ 2883 and λ 2895 was close to the excitation potential of the negative bands of carbon monoxide and of nitrogen, that is, close to 19 volts.

The discharge tube was now thoroughly outgassed and only traces of carbon remained on the walls of the tube. Oxygen alone was then introduced into the discharge tube at a pressure of 0.3 mm. of Hg; the color of the discharge was now a pale green. When the arc current was very low, about 2 milliamperes, the two bands λ 2883 and λ 2895 were very weak or absent compared to the other bands of carbon dioxide. The second negative bands of oxygen were very strongly developed, but the ultra-violet system of oxygen was not present. These ultra-violet bands of oxygen were excited whenever the arc current was very much increased and at the same time the intensity of the bands λ 2883 and λ 2895 was now greatly enhanced.

These bands therefore behave quite differently[†] from the other carbon dioxide bands when excited in mixtures of oxygen. They are always strongly excited whenever the conditions are such that the negative bands of carbon monoxide and of nitrogen and the ultra-violet bands of oxygen are excited. The ionization potential of carbon dioxide is 14.4 volts. If the final level of the bands λ 2883 and λ 2895 is assumed to be the normal state of the carbon dioxide ion, CO_2^+, then the initial level for these bands must be approximately 4.3 volts above the ionization potential. Therefore we obtain a value of 18.7 volts for the excitation potential of these bands, and this value is in close resonance with initial levels of the ultra-violet bands of oxygen. From the potential energy curves of oxygen as

given by Smyth,[9] it is seen, when the Franck-Condon principle is taken into account, that transition from the $^2\Pi$ excited level, the initial level of the ultra-violet bands, can take place in the range from 18.3 to 19.4 volts. The excited oxygen ions have therefore sufficient energy to simultaneously ionize and excite the normal carbon dioxide molecules. It is therefore very probable that the strong excitation of the bands λ 2883 and λ 2895

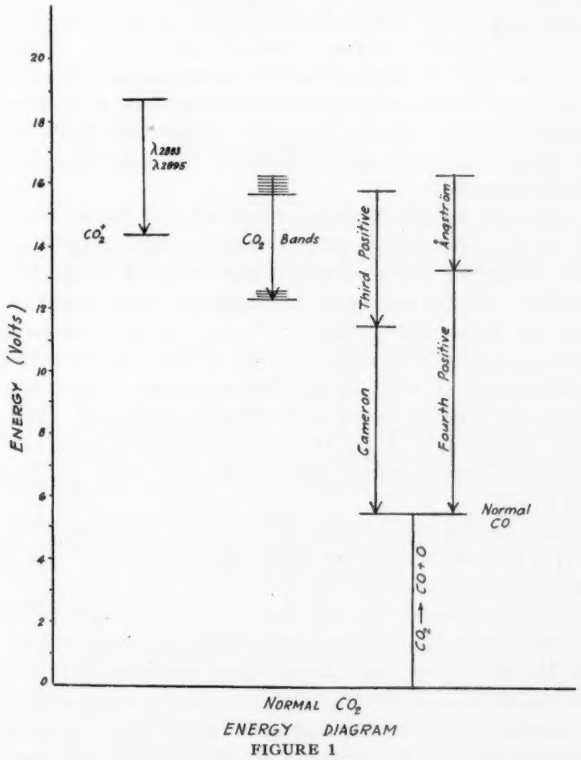


FIGURE 1

is caused by collisions of the second kind between the oxygen ions in these excited states and the carbon dioxide molecules, according to the process

$$\underset{18.3-19.4}{O_2^{+\prime}} + CO_2 \longrightarrow \underset{18.7}{CO_2^{+\prime}} + O_2$$

The emitter of the two bands λ 2883 and λ 2895 of carbon dioxide is therefore the carbon dioxide ion, CO_2^+; these bands have been fitted into the energy diagram in figure 1.

Several of the plates taken by Duffendack and Smith in their investigations on the excitation of diatomic molecules by impacts of the second kind were now examined. They found that the carbon dioxide spectrum was excited in He-O_2-CO and Ne-O_2-CO mixtures; the strength of the two bands λ 2883 and λ 2895 relative to the other carbon dioxide bands was carefully compared on their plates. In all the plates taken with the He-O_2-CO mixture the ultra-violet bands of oxygen were excited and also the negative system of carbon monoxide, but these systems were not excited in the Ne-O_2-CO mixture. Only in the spectra of the helium mixture are the bands λ 2883 and λ 2895 stronger than the other bands of carbon dioxide. We would expect this to be the case if these bands had an excitation potential close to that of the ultra-violet bands of oxygen; the effect however is small, due no doubt to the small concentration of oxygen in the mixture.

Some investigations had also been completed with mixtures of the rare gases and oxygen, with the intention of studying the ultra-violet system of oxygen. According to the above results if there had been a trace of carbon present then the two bands λ 2883 and λ 2895 should be excited along with the ultra-violet bands of oxygen. A series of photographs taken consecutively were examined to see if these bands were present. On the photographs of the A-O_2 and Ne-O_2 mixture, there was no trace of the bands of CO_2 nor of the ultra-violet bands of oxygen. On the other hand all the photographs of the He-O_2 spectrum showed the ultra-violet bands of oxygen well developed and at the same time the two bands λ 2883 and λ 2895.

These experimental observations are therefore conclusive; the bands λ 2883 and λ 2895 are due to the carbon dioxide ion CO_2^+, and have an excitation potential close to 18.7 volts.

Conclusion.—Reviewing the experimental observations we see that the λ 2883 and λ 2895 bands behave quite differently from the other bands of carbon dioxide. Under certain conditions we get a close relation between the carbon dioxide spectrum and the third positive and Ångström bands of carbon monoxide. This agrees with Smyth's observations and seems to show that this spectrum, except the bands λ 2883 and λ 2895 may be due to the neutral molecule. All the observations on the bands at λ 2883 and λ 2895 point to the fact that these bands are due to the carbon dioxide ion, CO_2^+, having an excitation potential about 18.7 volts, and their excitation in mixtures of carbon and oxygen seems to be due to collision of the second kind with excited oxygen molecular ions.

* Commonwealth Fellow.

† Prof. H. D. Smyth, who kindly read this manuscript stated in a private communication that he had also observed differences in the intensity variation of the λ 2883 and λ 2895 bands from that of the other CO_2 bands under certain excitation conditions.

[1] Fox, Duffendack and Barker, *Proc. Nat. Acad. Sci.*, **13**, 302 (1927).
[2] Kayser, *Handbuch Spectroskopie V*, **198**, 190–234 (1910).
[3] Hof, *Zeits. wiss. Phot.*, **14**, 83 (1914).
[4] Bair, *Astrophys. J.*, **52**, 301 (1920).
[5] Duffendack and Smith, *Phys. Rev.*, **34**, 68 (1929).
[6] Duncan, *Ibid.*, **34**, 1148 (1929).
[7] Smyth, *Ibid.*, **38**, 2000 (1931); **39**, 380 (1932).
[8] Henning, *Ann. d. Physik*, **13**, 599 (1932).
[9] Smyth, *Revs. Mod. Phys.*, **3**, 347 (1931).
[10] Schmid, *Phys. Rev.*, **41**, 732 (1932).
[11] Mulliken, *Ibid.*, **42**, 364 (1932).
[12] Jevons, *Phil. Mag.*, **47**, 586 (1924).
[13] Johnson, *Proc. Roy. Soc.*, **108**, 343 (1925).
[14] Fowler, *Ibid.*, **110**, 476 (1926).
[15] Emeleus and Emeleus, *Phil. Mag.*, **8**, 383 (1929).

GEOMETRY OF FOUR-COMPONENT SPINORS

BY OSWALD VEBLEN

INSTITUTE FOR ADVANCED STUDY

Communicated April 13, 1933

1. This note[1] is intended to be one of a series in which will be developed a geometrical theory of spinors which I have been putting forward in conversation and lectures at various times during the last three years. The basic idea is to use the Pluecker-Klein correspondence[2] between the lines of a projective 3-space and the points of a quadric in a 5-space in a manner inverse to that intended by Klein.

The equations of Pluecker are

$$X^{AB} = \frac{1}{2}(\psi^A \varphi^B - \varphi^A \psi^B) \quad (A, B = 1, 2, 3, 4). \tag{1.1}$$

Those of Klein are

$$\begin{aligned} X^4 &= X^{12} + X^{34} & X^5 &= -i(X^{12} - X^{34}) \\ X^2 &= X^{13} + X^{42} & X^3 &= -i(X^{13} - X^{42}) \\ X^0 &= X^{14} + X^{23} & X^1 &= -i(X^{14} - X^{23}) \end{aligned} \tag{1.2}$$

or, inversely written,

$$\begin{aligned} X^{12} &= \frac{1}{2}(X^4 + iX^5) & X^{34} &= \frac{1}{2}(X^4 - iX^5) \\ X^{13} &= \frac{1}{2}(X^2 + iX^3) & X^{24} &= \frac{1}{2}(X^2 - iX^3) \\ X^{14} &= \frac{1}{2}(X^0 + iX^1) & X^{23} &= \frac{1}{2}(X^0 - iX^1). \end{aligned} \tag{1.3}$$

All the quantities dealt with at this stage are ordinary complex numbers. The numbers $(\psi^1, \psi^2, \psi^3, \psi^4)$ may be interpreted as cartesian coördinates in an affine space S_4 of four (complex) dimensions which we are going to call the spin-space. Then all the points $(k\psi^1, k\psi^2, k\psi^3, k\psi^4)$ obtained from a given one by holding the ψ's fixed and giving k all values, constitute a straight line through the origin.

If P_3 is an arbitrary but fixed 3-space which does not pass through the origin of S_4 each of these lines cuts P_3 in one and but one point.[3] Thus the ψ's can be regarded as homogeneous coördinates for the points of P_3, $(\psi^1, \psi^2, \psi^3, \psi^4)$ and $(k\psi^1, k\psi^2, k\psi^3, k\psi^4)$ representing the same point.

Two points ψ and φ determine a plane of S_4 through themselves and the origin, consisting of all points whose coördinates are

$$\lambda\psi^A + \mu\varphi^A. \tag{1.4}$$

This plane cuts P_3 in a line, consisting of all points of P_3 having the homogeneous coördinates (1.4). The six numbers $(X^{12}, X^{13}, X^{14}, X^{34}, X^{42}, X^{23})$ given by (1.1) are at most multiplied through by a constant if ψ or φ is replaced in (1.1) by any point in the plane (1.4). Hence there is a set of numbers X^{AB} determined uniquely up to a factor of homogeneity by a given plane (1.4). It is easily verified that these X's satisfy the relation

$$X^{12}X^{34} + X^{13}X^{42} + X^{14}X^{23} = 0. \tag{1.5}$$

Inversely it is easily seen that any antisymmetric set of X's satisfying this relation determines a plane through the origin. Hence it is justifiable to regard them as the coördinates of the plane $0\psi\varphi$. Equally well we can regard them as the coördinates of the line joining the points ψ and φ of P_3; they are called the Pluecker coördinates of this line.

The six numbers $X^0, \ldots, X^5$ can be regarded as coördinates of points in an affine space A_6 of six (complex) dimensions. If P_5 is an arbitrary but fixed flat five-dimensional space not passing through the origin 0 of A_6, the X's can also be regarded as homogeneous coördinates of the point in which the line $0X$ meets P_5.

The equations (1.2) and (1.3) can be regarded as defining a transformation of cartesian coördinates in A_6 or of homogeneous coördinates in P_5. Under this transformation (1.5) becomes

$$(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 + (X^5)^2 = 0 \tag{1.6}$$

which is the equation of a cone through 0 in A_6. Equally well it is the equation of a non-degenerate quadric in P_5.

Thus the equations (1.1) determine a one-to-one reciprocal correspondence between the straight lines of P_3 and the points of the quadric in P_5. Equally well, they determine a correspondence between the planes through the origin of S_4 and the lines of the cone (1.5) or (1.6) in A_6.

2. The Pluecker-Klein representation was set up for the purpose of studying the system of lines in P_3 by means of their representation as points on the quadric in P_5. We propose to use it in an inverse manner.

We start with the four-dimensional space-time manifold of relativity theory, the points (events) being represented by coördinates (x^1, x^2, x^3, x^4). At each point there is a tangent space, T_4, in which the primary coördinates are (dx^1, dx^2, dx^3, dx^4). Each such tangent space T_4 will be identified with a sub-space of P_5 or A_6 and its representations by means of lines in P_3 or planes in S_4 will be used in order to provide a useful parameter representation.

This can be done in several ways according as T_4 is considered to be an affine, a projective or a conformal space. In all cases, however, T_4 is a real space. Therefore we have to look for suitable manifolds of four real dimensions in P_5 or A_6 (which have ten and twelve real dimensions, respectively). This leads to differential geometric applications of Juel and Segre's theory[4] of anticollineations, antiquadrics, etc.

Let us begin with the case which leads to a geometric interpretation of Fock's generalization[5] of the Dirac equations to general relativity. It is closely related to the corresponding two-component theory which stands in a similar relation to Weyl's treatment of the same problem.

The underlying space-time carries a quadratic differential form,

$$g_{ij}\, dx^i\, dx^j. \tag{2.1}$$

This determines a quadratic cone through the origin in each tangent space and we can introduce scalar coördinates by means of a real transformation

$$X^a = g^a_j\, dx^j \tag{2.2}$$

in such a way that (2.1) becomes

$$-(X^1)^2-(X^2)^2-(X^3)^2 + (X^4)^2. \tag{2.3}$$

Aside from the algebraic signs which appear in the quadratic form and the fact that $(X^0, \ldots, X^5)$ are supposed to be complex while $(X^1, \ldots, X^4)$ are real, this tangent space could be identified with the four-dimensional sub-space

$$X^0 = 0, \quad X^5 = 0$$

of A_6. These two obstacles are easily removed because the quadratic form in (1.6) can be transformed by a linear homogeneous transformation with complex coefficients into a sum of square terms affected with arbitrary signs.

We simply change to a coördinate system, X^σ, connected with the X^{AB} by the equations

$$X^{12} = \frac{1}{2}(X^5 + iX^0) \qquad X^{34} = \frac{1}{2}(X - iX^0)$$

$$X^{13} = \frac{1}{2}(X^3 + X^4) \qquad X^{42} = \frac{1}{2}(X^3 - X^4) \qquad (2.4)$$

$$X^{14} = \frac{1}{2}(X^1 + iX^2) \qquad X^{23} = \frac{1}{2}(X^1 - iX^2).$$

In this coördinate system the cone (1.5) has the equations,

$$(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 + (X^5)^2 = 0. \qquad (2.5)$$

Now consider the sub-space of A_6 consisting of those points which have real coördinates in the coördinate system, X^σ. The sub-space of this real space which is defined by the two equations,

$$X^0 = 0, \qquad X^5 = 0 \qquad (2.6)$$

can obviously be identified with the tangent space T_4 containing the cone (2.3).

Since the X's are now real it follows from (2.4) that

$$\begin{array}{l} X^{13} \text{ and } X^{42} \text{ are real} \\ X^{12} = X^{34*} \text{ and } X^{14} = X^{23*} \end{array} \qquad (2.7)$$

where we are using the * to indicate conjugate imaginaries.

The system of planes in the complex space S_4 (or of lines in the complex projective space P_3) defined by the conditions (2.7) is carried into itself by the transformation,

$$\begin{array}{llll} X^{12'} = X^{34*}, & X^{13'} = X^{13*}, & X^{14'} = X^{23*} \\ X^{34'} = X^{12*}, & X^{42'} = X^{42*}, & X^{23'} = X^{14*} \end{array} \qquad (2.8)$$

which is evidently of period two. This transformation of the planes of S_4 is effected by the point transformation $\psi \longrightarrow \varphi$ where

$$\begin{array}{l} \varphi^1 = -\psi^{3*} \\ \varphi^2 = -\psi^{4*} \\ \varphi^3 = \psi^{1*} \\ \varphi^4 = \psi^{2*} \end{array} \qquad (2.9)$$

in S_4. As a transformation of the complex space P_3 it is evidently of period two and is what is called an antiinvolution of the second kind.[4] Although it is of period four in S_4 we shall call it an antiinvolution in this space also.

The planes of the family (2.7) are now characterized as the set of planes through the origin joining each point ψ to its conjugate point in the anti-

involution (2.9). This means that we can replace φ by its expression according to (2.9) in the Pluecker formulas (1.1). Thus

$$\begin{aligned} X^{12} &= \frac{1}{2}(-\psi^1\psi^{4*} + \psi^2\psi^{3*}) & X^{34} &= \frac{1}{2}(\psi^3\psi^{2*} - \psi^4\psi^{1*}) \\ X^{13} &= \frac{1}{2}(\psi^1\psi^{1*} + \psi^3\psi^{3*}) & X^{42} &= \frac{1}{2}(-\psi^4\psi^{4*} - \psi^2\psi^{2*}) \\ X^{14} &= \frac{1}{2}(\psi^1\psi^{2*} + \psi^4\psi^{3*}) & X^{23} &= \frac{1}{2}(\psi^2\psi^{1*} + \psi^3\psi^{4*}). \end{aligned} \tag{2.10}$$

The X's given by these formulas represent one plane through each point ψ of S_4. No two planes of this family can have any point except the origin in common. For the point φ conjugate to ψ in the transformation (2.9) would then have to be $(\lambda\psi^1, \ldots, \lambda\psi^4)$ where λ is a constant, and the assumption $\lambda\psi^A = \varphi^A$, when substituted in (2.9) leads to a contradiction. This family of planes, defined either by the equations (2.7) or the equations (2.10), we shall call an *anticongruence* of planes.

Interpreting the ψ's and X's as homogeneous coördinates in P_3 the equations determine an anticongruence of lines, a system of lines of which one and only one passes through each point.

The equations (2.4) may be solved to give

$$\begin{aligned} X^0 &= -i(X^{12} - X^{34}), & X^5 &= (X^{12} + X^{34}), \\ X^2 &= -i(X^{14} - X^{23}), & X^1 &= (X^{14} + X^{23}), \\ X^4 &= (X^{13} - X^{42}), & X^3 &= (X^{13} + X^{42}). \end{aligned} \tag{2.11}$$

Thus we find

$$\begin{aligned} X^0 &= \frac{i}{2}(\psi^1\psi^{4*} - \psi^2\psi^{3*} + \psi^3\psi^{2*} - \psi^4\psi^{1*}) \\ X^1 &= \frac{1}{2}(\psi^1\psi^{2*} + \psi^4\psi^{3*} + \psi^2\psi^{1*} + \psi^3\psi^{4*}) \\ X^2 &= \frac{i}{2}(-\psi^1\psi^{2*} - \psi^4\psi^{3*} + \psi^2\psi^{1*} + \psi^3\psi^{4*}) \\ X^3 &= \frac{1}{2}(\psi^1\psi^{1*} - \psi^2\psi^{2*} + \psi^3\psi^{3*} - \psi^4\psi^{4*}) \\ X^4 &= \frac{1}{2}(\psi^1\psi^{1*} + \psi^2\psi^{2*} + \psi^3\psi^{3*} + \psi^4\psi^{4*}) \\ X^5 &= \frac{1}{2}(-\psi^1\psi^{4*} + \psi^2\psi^{3*} + \psi^3\psi^{2*} - \psi^4\psi^{1*}). \end{aligned} \tag{2.12}$$

The right-hand members of these equations are all Hermitian forms and therefore the X's are all real.

3. We may now interpret the equations (2.12) as follows. The ψ's are coördinates in a complex affine space S_4. The X's are coördinates in a real affine A_6. The equations (2.12) determine a correspondence $\psi \longrightarrow X$ such that each ψ corresponds to one X. The coördinates of X are, at most, multiplied by a constant if ψ is replaced by any point of the plane joining ψ to its conjugate in the antiinvolution (2.9). The correspondence $\psi \longrightarrow X$ therefore determines a one-to-one reciprocal correspondence between the anticongruence of planes (2.7) in S_4 and the lines of the cone (2.5) in A_6.

Obviously the ψ's may also be interpreted as homogeneous coördinates of points in a complex projective P_3 and the X's are homogeneous coördinates of points in a real projective P_5. The correspondence is then one between an anticongruence of lines in P_3 and a non-degenerate quadric in P_5.

The set of points in A_6 satisfying the two linear equations,

$$X^0 = 0, \quad X^5 = 0, \tag{3.1}$$

or the equivalent pair of equations

$$X^{12} = 0, \quad X^{34} = 0, \tag{3.2}$$

is a real flat affine space of four dimensions. This we identify with a tangent space T_4 of space-time. We also identify the coördinates X^1, X^2, X^3, X^4 in (2.11) with the scalar coördinates defined by (2.2). The four equations for $X^1, \ldots, X^4$ in (2.12) now give a parameter representation for the light-cone (2.3).

The equations (3.2) may be written,

$$\frac{\varphi^1}{\varphi^2} = \frac{\psi^1}{\psi^2}, \qquad \frac{\varphi^3}{\varphi^4} = \frac{\psi^3}{\psi^4}.$$

From this it follows that each plane of the spin-space S_4 which represents a point on the light-cone of T_4 contains a point of the plane

$$\psi^1 = \psi^2 = 0 \tag{3.3}$$

and a point of the plane

$$\psi^3 = \psi^4 = 0. \tag{3.4}$$

Indeed, each such plane joins a point $(\psi^1, \psi^2, 0, 0)$ to the point $(0, 0, \psi^{1*}, \psi^{2*})$.

When we introduce this specialization in the choice of the ψ's in (2.12) we obtain the parameter representation of the light-cone upon which the theory of two-component spinors is based (see the equations (4.2) of our paper on that subject in this volume of the PROCEEDINGS). This parameter

representation now appears as a correspondence between the points of the plane $\psi^3 = \psi^4 = 0$ and points of the light-cone.

4. This discussion brings to light a special sort of coördinate system for the tangent spaces T_4. First we pass from the dx coördinates to the scalar coördinates by the transformation

$$dx^i = g^i_a X^a \qquad (a = 1, 2, 3, 4). \tag{4.1}$$

Then we allow two extra coördinates X^0 and X^5 introduced for the sake of symmetry of notation and because they appear in a further reaching theory of which the present one is a special case. At present

$$X^0 = 0, \qquad X^5 = 0. \tag{4.2}$$

We shall use the last part of the Greek alphabet, beginning with σ, to denote indices ranging from 0 to 5. The equations (2.4) may now be written

$$X^{AB} = g^{AB}_\sigma X^\sigma \tag{4.3}$$

and we have at present,

$$X^{12} = X^{34} = 0. \tag{4.4}$$

Therefore the four complex numbers X^{13}, X^{14}, X^{34}, X^{23} may be thought of as coördinates of points in the real T_4. We have by definition,

$$X^{AB} = -X^{BA}. \tag{4.5}$$

The coefficients of (4.3) constitute six four-rowed antisymmetric matrices, as follows:

$$\ddot{g}_0 = \frac{1}{2}\begin{vmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{vmatrix} \qquad \ddot{g}_5 = \frac{1}{2}\begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

$$\ddot{g}_2 = \frac{1}{2}\begin{vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{vmatrix} \qquad \ddot{g}_1 = \frac{1}{2}\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} \tag{4.6}$$

$$\ddot{g}_4 = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} \qquad \ddot{g}_3 = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}$$

The equations (2.11) which are inverse to (2.4) may be written

$$X^\sigma = g^\sigma_{AB} X^{AB}. \tag{4.7}$$

The coefficients of these equations constitute the following six matrices:

$$g^{o}_{\cdot\cdot} = \frac{1}{2}\begin{Vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{Vmatrix} \qquad g^{5}_{\cdot\cdot} = \frac{1}{2}\begin{Vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{Vmatrix}$$

$$g^{2}_{\cdot\cdot} = \frac{1}{2}\begin{Vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{Vmatrix} \qquad g^{1}_{\cdot\cdot} = \frac{1}{2}\begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{Vmatrix} \tag{4.8}$$

$$g^{4}_{\cdot\cdot} = \frac{1}{2}\begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{Vmatrix} \qquad g^{3}_{\cdot\cdot} = \frac{1}{2}\begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}$$

Since (4.7) is inverse to (4.3) we must have

$$g^{\sigma}_{AB}\, g^{AB}_{\tau} = \delta^{\sigma}_{\tau} \tag{4.9}$$

and

$$g^{\sigma}_{AB}\, g^{CD}_{\sigma} = \frac{1}{2}\,\delta^{CD}_{AB}. \tag{4.10}$$

The points on the light-cone are expressible in the form

$$X^{AB} = \frac{1}{2}\,\delta^{AB}_{CD}\,\psi^{C}\,\varphi^{D} \tag{4.11}$$

where φ and ψ are related by (2.9) or, as we may write it

$$\varphi^{A} = S^{A}_{B}\;\psi^{B*} \tag{4.12}$$

with

$$S^{A}_{B} = \begin{Vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}. \tag{4.13}$$

The equations (2.11) now assume the form

$$X^{a} = g^{a}_{AB}\,\psi^{A}\,\varphi^{B} \tag{4.14}$$

and (2.12) becomes

$$X^{a} = g^{a}_{AC}\;\; S^{C}_{B}\;\;\psi^{A}\,\psi^{B*}. \tag{4.15}$$

5. The description in formulas of the geometric objects which we have been studying depends on four frames of reference.

(1) The coördinate system x in space-time and the associated coördinate system of differentials dx^{i} in the tangent spaces T_4.

(2) The frame of reference for scalar coördinates in the spaces T_4. These coördinates can be subjected to arbitrary Lorentz transformations.

(3) The spin-frame of reference, i.e., a homogeneous coördinate system in S_4. These coördinates can be replaced by any others obtained from them by a linear homogeneous transformation

$$\psi^A = T^A_B \psi^B \tag{5.1}$$

which leaves the formulas (2.9) for the fundamental antiinvolution and the fundamental planes (3.3) and (3.4) invariant. In case the formulas for each of the two planes are to remain separately invariant, the matrix of (5.1) must be of the form

$$T^A_B = \begin{Vmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a^* & b^* \\ 0 & 0 & c^* & d^* \end{Vmatrix} \tag{5.2}$$

where a, b, c, d are arbitrary and the * indicates conjugate imaginaries. If we allow the expression for the two planes to be interchanged we must also allow matrices of the form

$$T^A_B = \begin{Vmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ a^* & b^* & 0 & 0 \\ c^* & d^* & 0 & 0 \end{Vmatrix}. \tag{5.3}$$

The transformations (5.2) by themselves form a group and the transformations (5.2) and (5.3) together form a group.

(4) Finally, we have to allow for gauge transformations,

$$\bar{x}^0 = x^0 + \log \rho(x^1, \ldots, x^4), \tag{5.4}$$

or, more generally,

$$d\bar{x}^0 = dx^0 + v^0_i \, dx^i. \tag{5.5}$$

The coördinates ψ can be multiplied through by any factor of absolute value 1 without changing the point X represented by (2.12). This can be taken care of formally by introducing a factor

$$e^{ix^0}$$

where x^0 is an arbitrary real "gauge variable." The gauge transformation (5.4) changes the frame of reference for this variable in a way that depends on a point function in the underlying space-time. The more general gauge transformation (5.5) applied to

$$e^{i\int dx^0}$$

makes a change of gauge dependent on a path of integration in space-time.

We have to consider geometric objects analogous to tensors, having sets of components which obey the tensor law of transformation with respect to changes of all these frames of reference. Consider for example a set of 80 functions,

$$V^a_{A\alpha} = e^{Nx^0} f^a_{A\alpha}\ (x^1, \ldots, x^4). \tag{5.6}$$

The notation implies that under changes of coördinates and gauge they satisfy the transformation law

$$\bar{V}^a_{A\alpha} = V^a_{A\beta} \frac{\partial x^\beta}{\partial \bar{x}^\alpha} \qquad (\alpha, \beta = 0, 1, 2, 3, 4) \tag{5.7}$$

that under transformations of the scalar coördinates

$$\bar{X}^a = B^a_b\, X^b \tag{5.8}$$

they satisfy the law

$$\bar{V}^a_{A\alpha} = V^b_{A\alpha}\, B^a_b.$$

and that under spin transformations (5.1) they satisfy the law

$$V^a_{A\alpha} = V^a_{B\alpha}\, T^B_A\, t^M, \tag{5.9}$$

where

$$t^A_B\, T^B_C = \delta^A_B$$

and

$$t = |\, t^A_B\,|.$$

These functions are mixed components of a geometric object, which with respect to α is a projective covariant vector of index N, with respect to A a spinor of weight M, and with respect to a the scalar components of a contravariant vector. With respect to any one index we may think of V as having components which are the matrices obtained by letting the other indices take on all their values.

For the special problems which we have had in view above it is possible to interpret this definition in the sense that (5.8) are to be Lorentz transformations and (5.1) are subject to the conditions (5.2) and (5.3). It is probably better, however, in the long run, to let (5.8) and (5.1) be entirely arbitrary non-singular linear transformations. Particular subgroups of the linear group are then determined by specifying certain geometric objects which are left invariant. For example, the subgroups defined by the conditions (5.2) and (5.3) can be defined by means of the

spinors discussed in §6 below. The theory of spinors which arises on this basis is applicable to a wide class of geometries and not merely to the special case with which we started.

An important special case of a spinor is that of a contravariant spin vector of weight $^1/_4$ and index i. This has sets of four components

$$\psi^A = e^{ix^0} f^A_{(x)}$$

and the law of transformation,

$$\bar{\psi}^A = \psi^B T^A_B t^{\frac{1}{4}}. \tag{5.10}$$

For each point x of space-time such a spinor determines in the spin-space S_4 associated with x a line joining the origin to a point whose coördinates are ψ^A. There is an infinity of these points corresponding to the arbitrariness in the choice of x^0. They constitute a "chain" on the complex line 0ψ. These points all correspond under (2.12) to a single point X^a on the light-cone at x.

An arbitrary transformation (5.1) of spin coördinates subject to the conditions (5.2) or (5.3) brings about according to (5.10) a transformation on the coördinates ψ which is of determinant 1. This produces a Lorentz transformation on X^a and thus does not change[6] the point of the light-cone which is represented by ψ.

6. The equations (4.2) of T_4 as a subspace of A_6 define two fundamental skew symmetric spinors. These equations are

$$X^{12} - X^{34} = 0 \tag{6.1}$$

and

$$X^{12} + X^{34} = 0. \tag{6.2}$$

In S_4 (6.1) is the equation of a family of planes through the origin which we shall call a linear complex of planes (because it is the projection from the origin of a linear line complex in P_3). All planes of the family which pass through a point $(\varphi^1, \varphi^2, \varphi^3, \varphi^4)$ lie in a hyperplane, S_3, whose equation is

$$\begin{vmatrix} \psi^1 & \psi^2 \\ \varphi^1 & \varphi^2 \end{vmatrix} - \begin{vmatrix} \psi^3 & \psi^4 \\ \varphi^3 & \varphi^4 \end{vmatrix} = 0. \tag{6.3}$$

There is thus determined a correspondence in which each line joining the origin to a point φ^A corresponds to the hyperplane φ_A whose equation is (6.3). This correspondence between lines and planes through the origin will be called a *null-system* because it is the projection from the origin of what is ordinarily called a null-system in P_3.

We refer to the coefficients of this equation of a hyperplane as the "coordinates" of the hyperplane. If (6.3) is written in the form

$$\varphi_A \psi^A = 0 \tag{6.4}$$

we must have

$$\begin{aligned} \varphi_1 &= \varphi^2 \\ \varphi_2 &= -\varphi^1 \\ \varphi_3 &= -\varphi^4 \\ \varphi_4 &= \varphi^3. \end{aligned} \tag{6.5}$$

By reference to (4.6) and (4.8) we see that this relation may be written as

$$\varphi_B = \frac{1}{i} g^0_{AB} \varphi^A \text{ or as } \varphi^B = \frac{1}{i} g_0^{AB} \varphi_A. \tag{6.6}$$

In like manner (6.2) defines another linear complex of planes and a null-system,

$$\varphi_B = g^5_{AB} \varphi^A \text{ and } \varphi^B = g_5^{AB} \varphi_A. \tag{6.7}$$

The two linear complexes of planes have in common a linear congruence of planes, namely, the set of all planes joining the origin to a point of the plane (3.3) and a point of (3.4). The planes which are common to this congruence and to the antiinvolution (2.9) are the representatives of the lines of the light-cone (2.3).

7. Suppose now that we drop the condition (6.1) and retain (6.2). We then have by means of (2.12) a correspondence between the spin space S_4 and the real subspace T_5 of A_6 which is defined by the equation

$$X^5 = 0. \tag{7.1}$$

We may identify T_5 with the affine 5-space of all projective vectors of index 0 which is considered in the projective relativity[7] theory.

We may also consider $X^0, \ldots, X^4$ as homogeneous coördinates in the tangent space T_4 of space-time. In this tangent space we had a relation[8]

$$dx^i = \frac{Z^i}{\varphi_\alpha Z^\alpha} \tag{7.2}$$

between the fundamental coördinates dx and a set of coördinates Z which transform like projective vectors. The coördinates Z may be related to the coördinates X by equations of transformation

$$X^\alpha = \gamma^\alpha_\beta Z^\beta \tag{7.3}$$

in which $\gamma^0_\beta, \ldots, \gamma^5_\beta$ are five projective vectors so chosen that (7.2) shall

transform the generalized quadratic form of the projective relativity,

$$\gamma_{\alpha\beta} Z^{\alpha} Z^{\beta}, \tag{7.4}$$

into

$$(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2. \tag{7.5}$$

It is also possible to set up the relation between T_5 and the tangent space T_4 as it is done by Einstein and Mayer.[9]

In any case, the process which we have outlined sets up a correspondence between the points ψ of the complex spin-space S_4 and the points on the quadric,

$$\gamma_{\alpha\beta} Z^{\alpha} Z^{\beta} = 0, \tag{7.6}$$

in the tangent space T_4 of space-time. In the spin-space there is a family of planes common to the fundamental linear complex (6.2) and the fundamental antiinvolution (2.9) such that all points of one of these planes correspond to the same point of the quadric (7.5).

The correspondence between the points of S_4 and the points of the cone

$$(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 = 0 \tag{7.7}$$

in the "associated space" T_5 is such that if a point X is the correspondent of a point ψ of S_4 it is also the correspondent of each point

$$\lambda\psi + \mu\varphi$$

where

$$\lambda\lambda^* + \mu\mu^* = 1$$

and φ is the point given by (2.9).

8. Another application of the Pluecker-Klein representation is obtained if we interpret $(X^0, \ldots, X^5)$ as homogeneous coördinates in a projective 5-space P_5. Then the real points on the quadric (2.5) constitute a four-dimensional manifold which we can identify with a tangent space T_4 of a four-dimensional space after introducing a suitable set of "points at infinity" in T_4. The resulting geometry is a four-dimensional generalized conformal geometry.

If we adhere to the quadric (2.5) this is the conformal geometry associated with a Riemannian geometry of a definite quadratic differential form, i.e., one for which $g_{ij}\, dx^i\, dx^j$ has the signature $(+ + + +)$. To obtain a conformal geometry related to the indefinite quadratic form of the special relativity (2.3) we must replace (2.5) by

$$(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 - (X^5)^2 = 0. \tag{8.1}$$

A representation of this quadric is obtained by combining (1.1) with

$$X^{12} = \frac{1}{2}(X^4 + iX^5), \qquad X^{34} = \frac{1}{2}(X^4 - iX^5),$$
$$X^{13} = \frac{1}{2}(X^2 + iX^3), \qquad X^{42} = -\frac{1}{2}(X^2 - iX^3), \quad (8.2)$$
$$X^{14} = \frac{1}{2}(X^0 + iX^1), \qquad X^{23} = -\frac{1}{2}(X^0 - iX^1),$$

the inverse of which is

$$\begin{aligned} X^4 &= X^{12} + X^{34}, & X^5 &= -i(X^{12} - X^{34}), \\ X^2 &= X^{13} - X^{42}, & X^3 &= -i(X^{13} + X^{42}), \quad (8.3) \\ X^0 &= X^{14} - X^{23}, & X^1 &= -i(X^{14} + X^{23}). \end{aligned}$$

In the complex projective space P_3 in which points are represented by homogeneous coördinates $(\psi^1, \psi^2, \psi^3, \psi^4)$ let planes be denoted by homogeneous coördinates $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. Then, as is well known, the line of intersection of two planes σ and τ may be denoted by Pluecker coördinates

$$X_{AB} = \sigma_A \tau_B - \sigma_B \tau_A. \quad (8.4)$$

Indeed, if this line is the same as that given by (1.1) we have

$$X_{12} = \rho X^{34}, \quad X_{13} = \rho X^{42}, \quad X_{14} = \rho X^{23},$$
$$X_{34} = \rho X^{12}, \quad X_{42} = \rho X^{13}, \quad X_{23} = \rho X^{14},$$

where ρ is arbitrary. These equations may be written

$$X_{AB} = \rho \epsilon_{ABCD} X^{CD} \quad (8.5)$$

where ϵ is the alternating spinor of weight -1 such that $\epsilon_{1234} = 1$. Since $X^0, \ldots, X^5$ are to be real we must have

$$\begin{aligned} \rho^* X^{12} &= X^*_{12} & \rho^* X^{34} &= X^*_{34} \\ \rho^* X^{13} &= -X^*_{13} & \rho^* X^{42} &= -X^*_{42} \quad (8.6) \\ \rho^* X^{14} &= -X^*_{14} & \rho^* X^{23} &= -X^*_{23}. \end{aligned}$$

This transformation of the lines of P_3 is brought about by the *antipolarity*,

$$\begin{aligned} \sigma_1 &= \psi^{1*} \\ \sigma_2 &= \psi^{2*} \\ \sigma_3 &= -\psi^{3*} \quad (8.7) \\ \sigma_4 &= -\psi^{4*}. \end{aligned}$$

The set of points which lie on their antipolar planes satisfies the equation,

$$\psi^1\psi^* + \psi^2\psi^{2*} - \psi^3\psi^{3*} - \psi^4\psi^{4*} = 0. \quad (8.8)$$

They constitute what is called an *antiquadric*.

Thus the system of lines in P_3 which correspond to the points of the quad-

ric (8.1) in P_5 are the lines of the antiquadric (9.7). They are the lines joining pairs of points ψ and φ which satisfy (8.8) and

$$\psi^1\varphi^{1*} + \psi^2\varphi^{2*} - \psi^3\varphi^{3*} - \psi^4\varphi^{4*} = 0 \tag{8.9}$$

and

$$\varphi^1\varphi^{1*} + \varphi^2\varphi^{2*} - \varphi^3\varphi^{3*} - \varphi^4\varphi^{4*} = 0. \tag{8.10}$$

9. In each of the cases which have been described above the points of a quadric in a tangent space of space-time are represented by points of a spin-space. This correspondence is not in general one-to-one and its properties are determined by the signatures of the quadric and the nature of the tangent space, whether affine, projective or conformal. In each case it leads to a differential geometry of such spinors as are defined in §5, the special properties of the space in question being determined by the presence of certain invariant spinors.

For each case it is necessary to have an appropriate "spin connection." The properties of these connections can be worked out by the methods used in the paper on the geometry of two-component spinors in this volume of the PROCEEDINGS, and are, indeed, already partly worked out in the rapidly growing literature on the subject in physical journals. The suitable extensions of the process of covariant differentiation to spin quantities then come to light and have their application to the differential equations which connect the probability-of-a-particle theory with the corresponding field theory. These ideas I hope to expound more precisely in further notes.

[1] It first suggested itself as a possible basis for a geometrical interpretation of Eddington's theory of the interaction of electric charges, *Proc. Royal Soc.*, **A121**, 524 (1928), and was proposed to Prof. Schouten during his visit to America as a possible geometrical interpretation of the theory of spin-quantities which he was then developing. Cf. J. A. Schouten, "Dirac Equations in General Relativity," *Jour. Math. and Phys.*, **10**, 239 (1932).

[2] See the papers collected under the heading "Zur Liniengeometrie" in Vol. 1 of Klein's *Gesammelte Mathematische Abhandlungen*, Berlin, 1921.

[3] In order that this statement be correct also for the lines through the origin of S_4 and parallel to P_3, the latter must be a projective space.

[4] The theory of antiinvolutions, antiquadrics, etc., is chiefly due to C. Segre. A good exposition of it is to be found in E. Cartan, *Geometrie Projective Complexe*, Paris (1931).

[5] V. Fock, "Geometrisierung der Diracschen Theorie des Elektrons," *Zeitsch. Phys.*, **57**, 261 (1929).

[6] That a simple spinor should be of weight $^1/_4$ was found by J. A. Schouten from a different point of view.

[7] See O. Veblen, *Projektive Relativitätstheorie*, Berlin (1933).

[8] *Ibid.*, p. 12. In this book X's are used for the coördinates which we are now denoting by Z's.

[9] Einstein and Mayer, *Einheitliche Theorie von Gravitation und Elektrizität*, *S.-B. Preuss. Akad. Wiss.*, 541–557 (1931). See also the last chapter of the book referred to in Note 7.

CONTRIBUTION TO THE THEORY OF SCATTERING BY A FORCE CENTER

BY E. H. KENNARD

DEPARTMENT OF PHYSICS, CORNELL UNIVERSITY

Communicated March 20, 1933

In treating by wave-mechanics the scattering of a particle by a force center, it is often convenient to use characteristic functions which represent plane waves falling upon the center and being scattered by it, and in some cases it is important to know that these solutions form, as do the incident waves alone, an orthogonal set of characteristic functions. This property was established by Sommerfeld for the special functions used in his treatment of the emission of continuous x-rays. We shall show that a general proof covering all cases can also be given.

The line of argument is suggested by considering two wave-packets which start at infinity and pass over the force center simultaneously. Initially two such packets are "orthogonal" simply because they do not overlap; this orthogonality is then maintained permanently by the Schrödinger time equation; and if the packets are approximately "monochromatic" they become, as they pass over the center, practically the same as two of the characteristic functions in question.

To put the argument into mathematical form, let

$$\psi(x, y, z, \mu) = C(\mu)e^{i\mu\cdot r} + \eta(x, y, z, \mu) \tag{1}$$

be such a scattered-wave solution of the equation

$$\Delta\psi + (\mu^2 - aV)\psi = 0, \tag{2}$$

where V is the potential function of the force center, r = radius vector and μ is proportional to the vector momentum of the impinging particle.[1] We shall assume that, as is usual in practical problems,

$$\eta = F(r, \theta, \varphi, \mu)e^{i\mu r}/r \tag{3}$$

where F becomes at infinity a definite function of the polar angles θ, φ and of μ; further restrictions upon η will be laid down as needed, but they are all met at least if F has bounded and continuous derivatives relative to all arguments as far as the third order and if those derivatives which involve r twice vanish at infinity as $1/r^2$ uniformly in the other arguments.

Orthogonality now requires that

$$I = \int \psi^*(\mu_1)d\tau \int_{\delta\pi} \psi(\mu)d\pi = 0 \tag{4}$$

unless μ_1 lies within the region $\delta\pi$ in μ-space ($d\tau$ = element of volume in coördinate space). The most interesting case is that in which μ_1 differs

only in direction from certain values of μ lying in $\delta\pi$, for in such a case with a symmetrical force center η/F is the same for these values and one might expect the vanishing of I to be prevented by the term in $\eta^*(\mu_1)\eta(\mu)$.

To show that (4) nevertheless holds even in such cases, let us generalize the integral I into the equivalent of a moving packet:

$$J = \int e^{-i\beta\mu_1^2}\psi^*(\mu_1)d\tau \int_{\delta\pi} e^{i\beta\mu^2}\psi(\mu)d\pi. \tag{5}$$

Differentiating and substituting the value of $\mu^2\psi(\mu)$ from (2), one finds, after an integration by parts, that $dJ/d\beta = 0$. For $\beta = 0$, $J = I$; hence (4) will be established if we can show that, unless μ_1 lies in $\delta\pi$, $J \longrightarrow 0$ as $\beta \longrightarrow \infty$.

Now substitution of ψ from (1) in (5) gives a first term which is independent of η and by ordinary Fourier theory vanishes for any value of β unless μ_1 lies in $\delta\pi$. That part of each of the remaining terms which arises from integration out to any given finite distance a from the origin then vanishes as $\beta \longrightarrow \infty$ by the Riemann-Lebesgue lemma:

$$\lim_{b \longrightarrow \infty} \int_{\mu'}^{\mu''} f(\mu)e^{ib\mu}d\mu = 0 \tag{6}$$

provided $\int_{\mu'}^{\mu''} f(\mu)d\mu$ exists properly or converges absolutely, from which, by an easy change of variable,

$$\lim_{\beta \longrightarrow \infty} \int_{\mu'}^{\mu''} f(\mu)e^{i\beta\mu^2}\, d\mu = 0 \tag{7}$$

also. It suffices then to show further merely that the parts of these three remaining terms arising from space beyond $r = a$ vanish as $a \longrightarrow \infty$ independently of β, so that we can first choose a large enough to make these outlying parts less than any assigned quantity and then, without disturbing this condition, allow β to become infinite.

Complications are encountered at this point, however, due to the infinite range of r; convergence of the integral at infinity depends essentially upon the integrations over θ and μ, and the most feasible procedure seems to be to integrate several times by parts over the latter variables and then patiently discuss the resulting expressions. We shall give details only for the most complicated of the remaining three terms in J, whose part outside of $r = a$ is, from (5), (1), (3),

$$J_{2a} = e^{-i\beta\mu_1^2} \int_{r=a}^{r=\infty} \frac{1}{r} F^*(r, \theta, \varphi, \mu_1)e^{-i\mu_1 r}\, d\tau \int_{\delta\pi} C(\mu)e^{i(\mu\cdot r + \beta\mu^2)}d\pi.$$

Here it is convenient to suppose the space integration to be carried out first and to take as the axis of polars the direction of μ, so that $\mu\cdot r = \mu r \cos\theta$; then, writing $-\cos\theta = \zeta$,

$$J_{2a} = e^{-i\beta\mu_1^2} \lim_{b \longrightarrow \infty} \int_{\delta\pi} C(\mu) e^{i\beta\mu^2} d\pi \int_a^b r e^{-i\mu_1 r}\, dr \int_{-1}^{1} \int_0^{2\pi} F^* e^{-i\mu r\zeta}\, d\varphi d\zeta,$$

where F is now a function of μ as well as of μ_1 because the position of the axis for θ and φ varies with μ. Let $\gamma(r, \zeta, \mu)$ denote the result of integrating CF^* over φ and the angles included in $\delta\pi$; then, after three integrations by parts relative to ζ,

$$J_{2a} = e^{-i\beta\mu_1^2} \lim_{b \longrightarrow \infty} \int_{\delta\mu} e^{i\beta\mu^2} d\mu \int_a^b e^{-i\mu_1 r} dr \left\{ \frac{i}{\mu} \gamma(r, 1, \mu) e^{-i\mu r} - \frac{i}{\mu} \gamma(r, -1, \mu) e^{i\mu r} + \frac{1}{\mu^2 r} \frac{\partial \gamma}{\partial \zeta} (r, 1, \mu) e^{-i\mu r} - \frac{1}{\mu^2 r} \frac{\partial \gamma}{\partial \zeta} (r, -1, \mu) e^{i\mu r} + \frac{i}{\mu^3 r^2} \left[\frac{\partial^2 \gamma}{\partial \zeta^2} e^{-i\mu r\zeta} \Big/_{\zeta=-1}^{\zeta=1} - \int_{-1}^{1} \frac{\partial^3 \gamma}{\partial \zeta^3} e^{-i\mu r\zeta} d\zeta \right] \right\}. \quad (8)$$

Now in the first term on the right in (8) the integral in dr after an integration by parts becomes

$$-\frac{1}{\mu(\mu + \mu_1)} \left[\gamma(b) e^{-i(\mu+\mu_1)b} - \gamma(a) e^{-i(\mu+\mu_1)a} - \int_a^b \frac{\partial \gamma}{\partial r} e^{-i(\mu+\mu_1)r} dr \right],$$

of which the second term, being independent of b, can be associated with that part of J which arises from $r < a$ and vanishes with it as $\beta \longrightarrow \infty$. The third term we shall dispose of by assuming F to be so well-behaved at infinity that $\int_a^\infty (\partial\gamma/\partial r) e^{-i(\mu+\mu_1)r} dr$ converges uniformly in μ and therefore $\int_a^b (\partial\gamma/\partial r) e^{-i(\mu+\mu_1)r} dr$ vanishes uniformly in μ as $a \longrightarrow \infty$. This condition is certainly met if we can integrate once more by parts with respect to r and if $\partial^2\gamma/\partial r^2$ vanishes at infinity as $1/r^2$, and does this uniformly in μ, since for a function $f(r,\mu)$

$$\left| \int_a^b \frac{f(r,\mu) dr}{(c_1 + r)(c_2 + r)} \right| < \frac{M}{a}, \quad b > a > 0, \quad c_1 \geqq 0, \quad c_2 \geqq 0, \quad (9)$$

M a constant, provided $f(r)$ is bounded toward infinity uniformly in μ. Finally, the first term in the last bracket makes a contribution to the integral in μ in (8) that can be written, after a partial integration,

$$\frac{i}{\mu(\mu + \mu_1)} \frac{\gamma^*(b, 1, \mu)}{2\beta\mu - b} e^{i\beta\mu^2 - i(\mu+\mu_1)b} \Big/_{\mu'}^{\mu''} - \int_{\delta\mu} e^{i\beta\mu^2 - i(\mu+\mu_1)b}$$

$$\frac{\partial}{\partial\mu}\left(\frac{i}{\mu(\mu+\mu_1)}\,\frac{\gamma^*(b,1,\mu)}{2\beta\mu-b}\right)d\mu,$$

which vanishes as $b \longrightarrow \infty$ provided γ and $\partial\gamma/\partial\mu$ are bounded uniformly in μ.

In the second term on the right in (8) we must integrate with respect to μ instead of r because $\mu-\mu_1$ in the exponent may vanish; the result for the double integral is:

$$\int_a^b e^{-i\mu_1 r}\,dr\left\{\left[-\frac{\gamma}{\mu Z}+\frac{1}{iZ}\,\frac{\partial}{\partial\mu}\,\frac{\gamma}{\mu Z}\right]e^{i(\beta\mu^2+\mu r)}\Big|_{\mu'}^{\mu''}-\int_{\delta\mu}\frac{1}{i}\,\frac{\partial}{\partial\mu}\left(\frac{1}{Z}\,\frac{\partial}{\partial\mu}\,\frac{\gamma}{\mu Z}\right)e^{i(\beta\mu^2+\mu r)}\,d\mu\right\}$$

where $\gamma = \gamma(r, -1, \mu)$ and $Z = 2\beta\mu + r$. Here we dispose of the first term by assuming that $\int_a^\infty e^{i\lambda r}\gamma/(c+r)dr$ converges for $\lambda > 0$ and uniformly for any $c \geqq 0$ and so vanishes as $a \longrightarrow \infty$ independently of ρ (the condition $\lambda > 0$ excludes the possibility that $\mu_1 = \mu'$ or μ'', but this restriction is probably unimportant). The last terms then yield nothing because of (9), provided we assume γ, $\partial\gamma/\partial\mu$ and $\partial^2\gamma/\partial\mu^2$ all to be bounded at infinity uniformly in μ.

Finally, the third term in (8) yields nothing important provided $\int_a^\infty(\partial\gamma/r\partial\zeta)e^{-i(\mu+\mu_1)r}\,dr$ also converges and vanishes as $a \longrightarrow \infty$ uniformly in μ; and an integration in μ converts the fourth term into expressions of order $1/r^2$, which then, along with the remaining terms in (8), by (9) give nothing provided $\partial\gamma/\partial\zeta$, $\partial^2\gamma/\partial\zeta^2$ and $\partial^3\gamma/\partial\zeta^3$ are all bounded at infinity independently of μ and ζ.

This completes the proof for J_2. The remaining terms in J can be handled in a similar manner; and the desired proof of orthogonality is then complete.

[1] It has been impracticable to indicate in the notation when μ stands for a vector and when for a scalar quantity; the experienced reader will have no difficulty in inferring in each case which is meant.

THE INFLUENCE OF INTENSITY, COLOR AND RETINAL LOCATION ON THE FUSION FREQUENCY OF INTERMITTENT ILLUMINATION

BY SELIG HECHT AND CORNELIS D. VERRIJP*

LABORATORY OF BIOPHYSICS, COLUMBIA UNIVERSITY

Communicated April 3, 1933

1. *Nature of Problem.*—The flickering sensation produced by regularly interrupted illumination disappears when the frequency of interruptions is made sufficiently high. Under controlled conditions the determination of this critical fusion frequency may be made with considerable accuracy; its precise value, however, depends on a variety of conditions, of which the most effective is the intensity of the illumination.

The relation between these two variables is not wholly clear at present. Though the dependence of the critical frequency on illumination was recognized over one hundred years ago, it was only forty years ago that Ferry[1] formulated what has since become known as the Ferry-Porter law, namely, that the critical frequency is proportional to the logarithm of the illumination intensity. Ferry's published measurements distinctly do not support this generalization, but the later data of Porter[2] do. Porter, however, found that when the critical fusion frequency—as cycles of light and dark per second—is plotted against the logarithm of the intensity the data fall on *two* straight lines instead of one. The two lines intersect at an illumination of about 0.25 meter candles, and the slope of the lower is 1.56 while that of the upper is 12.4. These findings were corroborated by Ives[3] in 1912. Ives's data for different parts of the spectrum show a dual logarithmic relation similar to that for white light. However, the slope of the straight lines and their point of intersection seem to vary with the wave-length of the light, the upper and lower limbs of the relationship varying in different ways. In addition Ives found the extraordinary fact that for blue light, the lower line becomes horizontal.

These peculiarities are difficult to reconcile with the obvious interpretation of Porter's data which is generally given in terms of the Duplicity Theory, that is, that the lower limb describes the function of the rods while the upper limb describes the function of the cones. This difficulty has been emphasized by Allen[4] whose flicker studies in general confirm Porter and Ives, though differing from them in particulars. Thus Allen draws through his measurements about five short straight lines of different slope, instead of the two drawn by Porter and by Ives.

In our estimation, the data presented by Allen do not justify such treatment. The points seem to lie on a continuously curving line, and deviate from it no more than would be expected when special precautions are

omitted with regard to rigid fixation. This judgment is confirmed by the recent work of Lythgoe and Tansley[5] and of Granit and Harper[6] which gives no support to Allen's multiplicity of straight lines. Using controlled fixation, Lythgoe and Tansley confirm in general the logarithmic relation but attach no importance to its strict formulation.

One striking thing appears in the work of Lythgoe and Tansley though they do not recognize its significance. Ives had found that for blue light the lower limb of his data is horizontal, and in this he had been confirmed by Allen.[7] This seemed a special property of blue light. However, Lythgoe and Tansley have recorded that when measurements are made with a retinal area 10° from the center of the eye the lower portion of the data tends to be horizontal even for white light.

Our determination to study the phenomenon of flicker was prompted first, by the confused state of the situation in its relation to rod and cone functions; and second, by the fact that none of the existing measurements cover a range of intensities sufficient to define the relation between critical frequency and intensity over the functional range of the eye. We therefore measured this relation for different portions of the retina, for different colors, for as large a range of illuminations as possible, and under such conditions of fixation and surrounding illumination as to render the data reproducible and definitive.

2. *Apparatus and Method.*—The arrangement of the apparatus is shown diagrammatically in figure 1. The source of light is a concentrated-filament, 500-watt, projection Mazda lamp, running on direct current from storage cells. The lamp is in a rectangular housing having a circular opening in each of two adjacent walls. The openings are covered with ground glass, and serve as two secondary sources of illumination.

The light from one is deflected 90° in its path, by a totally reflecting prism, and focussed by a lens into the plane of a rotating, sectored wheel. The diverging light then passes through a hole in the silvering of a photometer cube, immediately after which it is focussed by a lens on to the exit pupil. Between this last lens and the exit pupil there are (*a*) places for filters, of which we used both neutral and monochromatic, (*b*) a neutral, balanced, gelatine wedge and (*c*) a very thin slip of glass, tilted so as to reflect a red fixation point into the eye looking through the exit pupil.

The light from the other ground glass of the lamp house passes through an identical optical system and ultimately impinges on the photometer cube, where it is reflected from the silvered diagonal face, through the lens, filters, wedge and glass slip, on to the exit pupil.

This exit pupil is a circular opening 1.8 mm. in diameter, and constitutes the artificial observation pupil. An eye looking through it sees the photometer cube through the wedge, balancer, filters and lens, and sees it bounded by the circular edge of the lens. The visual field is thus a circular

area 2 degrees in diameter illuminated with intermittent light, and surrounded by a circular field 10 degrees in diameter illuminated with continuous light. The brightness of the whole field is uniform; that is, the brightness of the surrounding field is the same as the brightness of the test field when the sectored wheel is going well beyond the critical frequency of flicker, and the central field appears continuously illuminated.

The intensity of the light coming through the exit pupil is varied by means of two neutral filters transmitting approximately 1/100 and

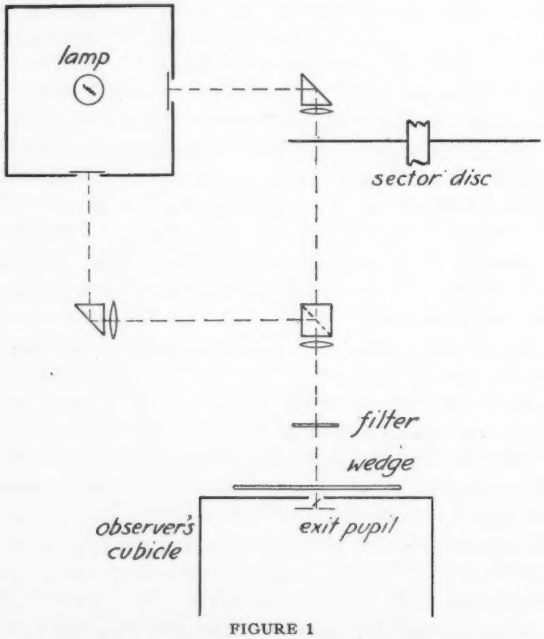


FIGURE 1
A simplified, diagrammatic plan of the optical arrangements used in the measurements of critical frequency.

1/10,000 of the light, and a neutral wedge with a transmission range of 1 to 1000. This combination covers easily and accurately a range of illumination between 1 and 50,000,000 units. The color of the light is varied by placing Wratten Monochromatic Filters, numbers 70 to 76, in the path of the light. The speed of the sector disc is controlled by rheostats and is measured by timing the contacts made by the rotating shaft through a system of electrical relays which may be made to record automatically.

Each series of measurements was begun at the lowest intensities. Therefore the subject was well dark-adapted before. Sufficient time was allowed at each measurement for the retina to become adapted to the particular intensity being measured, a procedure we found to be very important for the reproducibility of the data. All the readings recorded are for the right eye of S. H. and for the right eye of C. D. V.

3. *Foveal Measurements.*—The data which we secured fall into several groups, depending on the ideas which urged us to make them. The measurements of Porter, with their division into two portions, made sense in terms of a separation of rod and cone function in flicker. If, in spite of subsequent confusion, this separation of rod and cone function is real, it should be possible to get a complete cone curve by confining the measurements to the rod-free area of the fovea. Our first measurements were therefore made with strictly central fixation, which was maintained at the low intensities even when fixation normally would wander to the periphery. Because the flickering area is 2° in diameter, central fixation makes it fall on a retinal region which is practically free of rods.[8]

The data are given in figures 2 and 3. Each point of the foveal readings is the average of about 12 readings for S. H. and about 30 for C. D. V. It is clear that for the fovea there is one continuous relation between critical frequency and the logarithm of the intensity. The relationship, however, is not rectilinear, but distinctly sigmoid, the *S*-shape being rather drawn out. In the range of intensities lying between about 0.03 photon and 30 photons, the data lie with reasonable precision on a straight line. In this respect we can confirm Porter, Ives and the other workers. Below 0.03 photon the critical frequency continues to decrease as log I decreases, forming a gentle curve and stopping fairly abruptly when with central fixation the field appears uniform even when the test area is extinguished.

At the highest intensities the relation between critical frequency and log I rapidly ceases to be linear. As the intensity is raised a maximum critical frequency is soon reached, beyond which a further increase in intensity results in no further increase in critical frequency; rather it results in a decrease. The maximum critical frequency comes at about 200 photons for S. H. and at about 500 photons for C. D. V. The value of the critical frequency at this maximum is 53 cycles per second for C. D. V. and 45 cycles per second for S. H. With a further increase in the intensity, the critical frequency distinctly decreases.

The slope of the middle portion of the data is 11.1 for C. D. V. and 11.0 for S. H. It is important to point out that this is the same magnitude of slope found by Porter and by Ives for that portion of their data which has been generally considered as representing the cone function. In our measurements we have isolated that function so that it appears in complete form, and we have identified it anatomically with a region of the eye which

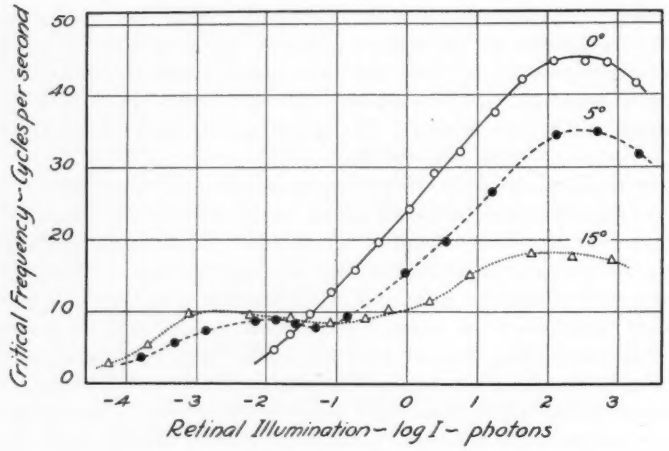


FIGURE 2

Data for S. H., showing relation between critical frequency and log I for white light for three different retinal locations: at the fovea, and at 5° and 15° above the fovea.

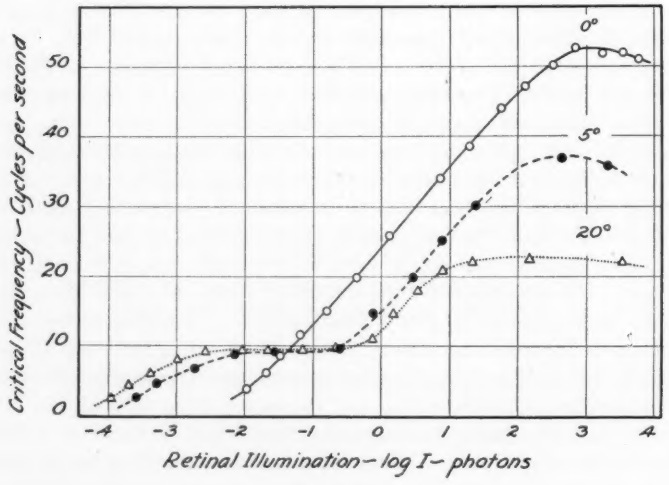


FIGURE 3

Data for C. D. V. showing relation between critical frequency and log I for white light for three retinal locations: at the fovea, and at 5° and 20° above the fovea.

contains only cones. There seems little doubt that the data represent the flicker function of the cones.

4. *Peripheral Measurements.*—The correctness of this conclusion becomes even more apparent when the measurements are made with regions of the retina outside the fovea centralis. We measured the critical frequency for white light with the same set-up as before but with fixation at 5° above the center. The data are given in figure 2 for S. H. and in figure 3 for C. D. V.

The measurements clearly fall into two parts. The first is at low intensities, where the critical frequency first rises with log I and then reaches a maximum which is maintained approximately constant for 1.25 logarithmic units. The total intensity range covered by this rise and plateau is about 3.25 logarithmic units. The second part also begins with a rise in critical frequency as log I increases, and also terminates when the critical frequency reaches a maximum, and then declines. The intensity range covered by the second part is about 4 logarithmic units.

These results are so striking that we wished to be certain of their general validity over the retina. We therefore measured the relation between critical frequency and illumination for the same test area and surround as before, but placed 5° peripherally in the four principal directions: up, down, nasal, temporal. The data for C. D. V. are given in figure 4, where each point is the average of two or three concordant readings. It is apparent that the essential phenomenon recorded is a general one, since the data all show the same division into two parts, with a rise and a plateau for each part.

Of the two, the part at the higher illuminations resembles the foveal curve in appearance, and has practically the same slope. Most likely, therefore, this portion represents the behavior of the cones. Moreover, the portion at low illuminations is a distinct and complete relationship, and does not appear in measurements made in the rod-free area. The obvious conclusion here is that the rise and the plateau at low illuminations represent the function of the rods.

Measurements with the test field still farther out in the periphery confirm and extend these findings. The data for a test area placed at 15° above the center are given in figure 2. The data for a similar area placed at 20° above the center are given in figure 3. The data show the same division into two parts, each with a rise of critical frequency versus log I and subsequent plateau as do the data already given for a 5° peripheral displacement.

The plateau for the 15° and 20° off-center measurements is about 0.75 log units longer than for the 5° off-center data. This is because at 15° and 20° the low values of the critical frequency occur at lower intensities and the high values occur at higher intensities than at 5° off-center. Thus the rod system becomes more sensitive and the cone system less sensitive

as the measuring area moves from the center farther into the periphery. This is in keeping with the anatomical increase in number of rods and the converse decrease in cones as one proceeds along the retina from the center toward the periphery.

5. *Color.*—The differential separation of cone function and rod function along the intensity axis may be accomplished without change of retinal position if one uses differently colored lights. Figure 4 gives a combined plot of the spectral sensibility distributions of the cones and of the rods. The former are the data of Hyde, Forsythe and Cady[9] describing the relative energy for different parts of the spectrum required for a constant brightness effect for the fovea at high illuminations, expressing the cone function. The latter are the similar data of Hecht and Williams[10] for the

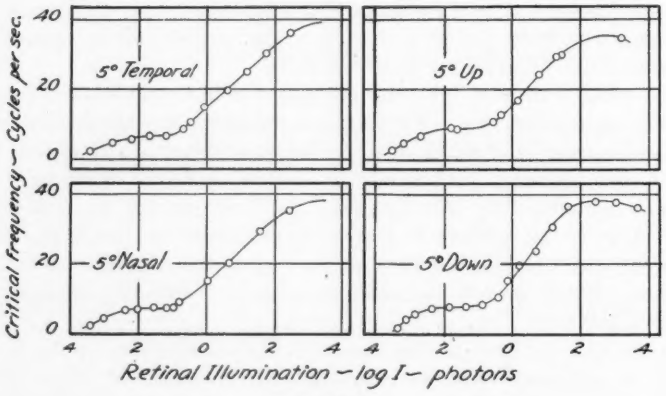


FIGURE 4

Data for C. D. V. showing relation between critical frequency and log I for white light for 5° off-center in the four principal retinal directions.

periphery at nearly threshold illuminations when there is no color visible, thus expressing the function of the cones. The two curves are arbitrarily made almost to coincide in the red region of the spectrum, since it is well known that in that region the color threshold is only slightly above or almost coincident with the colorless threshold.

Obviously if the relation between intensity and critical frequency is measured for a peripheral region containing rods and cones, the separation between the rod portion and the cone portion of the data should depend on the wave-length. In the red, the two portions should be only slightly, if at all, separated on the log I axis, whereas as the wave-length decreases toward the violet the separation should continue to become greater and greater.

To test this we made measurements with the field placed 5° above the center, using different parts of the spectrum isolated by Wratten monochromatic filters. The resulting data for C. D. V. are given in figure 5. The wave-lengths in the inset give in each case the center of the trans-

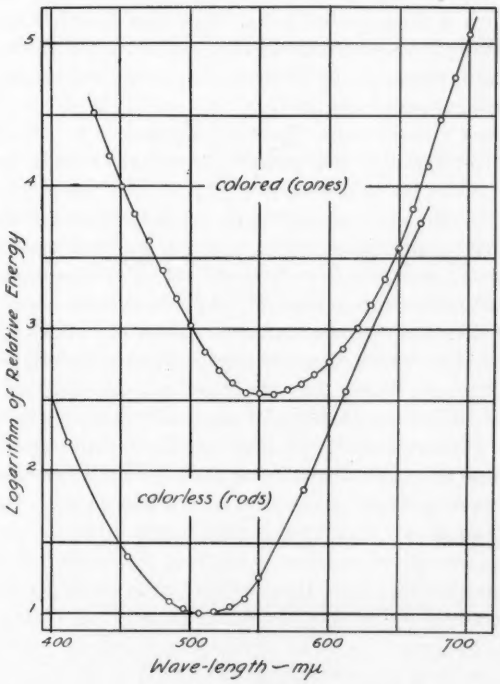


FIGURE 5

Comparison of rod and cone sensibility distributions in the spectrum, taken from the data of Hyde, Forsythe and Cady, and of Hecht and Williams. The curves are each accurately drawn from their separate data. Their vertical separation, however, has been arbitrarily arranged so that they are nearly coincident in the red; this is a graphic expression of the fact that the colorless and color thresholds of the eye are nearly coincident in the red.

mission band of the filter. The data for all colors at high intensities show a relationship between critical frequency and log I similar to that found for white light under similar conditions. In figure 5 we have plotted the data for all the colors in such a way that this high intensity portion is superimposed on the similar portion of the white curve for 5° of figure 3.

The slopes of all the colors for this portion are not identical, but they are near enough so that this procedure is possible. The different positions of the low intensity portions of the data are thus at once apparent. They show that the degree of separation of the two parts of the data for each wave-length is a function of the wave-length, and that the magnitude of the separation is of the expected order. One may therefore conclude that the two portions of the data correspond to the two systems in the retina, the low intensity portion to the function of the rods, and the high intensity portion to the function of the cones.

6. *Theoretical Formulation.*—The form of the data for the two receptor systems seems well enough defined to risk a tentative theoretical formulation of them in terms of known and hypothetical properties of vision. The outside stimulus alternates abruptly between light and dark. When this alternation occurs infrequently, the full difference is sharply perceptible in sensation. The more frequent the alternation, the less is the difference in sharpness and in intensity between the successive sensations; and when the frequency of alternation is sufficiently rapid the difference between the successive sensations vanishes and the outside fluctuating light appears continuous. Where does the significant transformation occur?

The work of Adrian and Matthews[11] shows unmistakably that this transformation is accomplished in the retina. They found that in the eel's eye the total rate of impulses proceeding along the optic nerve becomes uniform and shows no periodic changes for as low a frequency of intermittent retinal illumination as 5 cycles per second at low intensities, even though the nerve is shown to be capable of carrying a sharply defined series of impulse groups occurring three times as frequently at higher illuminations. Such a result would follow whether flicker depends on regular variations in the number of elements sending impulses or on regular variations in the frequency of discharge of continuously functional elements. In any case, flicker is essentially a retinal problem.

The retina is a complicated structure, and not enough is known about it to furnish the material for an adequate formulation of so complex a phenomenon as flicker. However, no matter what else occurs in the retina, the very first step in vision must involve a photochemical change having certain fairly well-defined characteristics. Since the products of this photochemical change serve only to *start* the complicated train of events which finally yield a series of impulses in the optic nerve, it cannot be expected that the behavior of the photochemical system alone will yield a complete description of the receptor process. Nevertheless, it is necessary to point out that this very first step in its relation to intermittent illumination already shows many of the essentially quantitative properties of flicker.

The familiar conception[12] of the first step in the photoreceptor process regards it is as a reversible photochemical reaction in which a sensitive sub-

stance is changed by the light into photoproducts which recombine under certain conditions to form the original substance. The velocity of the process under the influence of light is

$$\frac{dx}{dt} = k_1 I(a - x) - k_2 x^2 \qquad (1)$$

where I is the intensity, a the initial concentration of sensitive material and x the concentration of photoproducts. In the absence of light, only the "dark," regenerating reaction goes, and the equation

$$\frac{dx}{dt} = k_2 x^2 \qquad (2)$$

gives the rate at which it forms the photosensitive material.

In intermittent illumination these two reactions alternate rapidly, and at the disappearance of flicker, they form a steady state in which what has been decomposed during the light period is regenerated during the dark period. The time occupied by each light or dark exposure is Δt, and is very short when flicker disappears. Therefore the velocity dx/dt may be considered constant, and equal to $\Delta x/\Delta t$, where Δx is the change in concentration of photoproducts occurring between the beginning and the end of each exposure. Since the light and dark periods in these experiments are of equal duration, these two velocities are equal. On equating them we get

$$\frac{KI}{2} = \frac{x^2}{a - x} \qquad (3)$$

where $K = k_1/k_2$. This is Talbot's law for equal light and dark periods.[13]

For the critical disappearance of flicker it is supposed that $\Delta x = c'$, that is, that the fluctuation Δx in the chemical concentration of photoproducts is constant at a value c' which is just too small to cause the physiological change corresponding to a perceptible change in the sensation of brightness. The critical frequency is n cycles of light and dark flashes per second. Thus $n = 1/2\Delta t$. Substituting these values of Δx and of Δt in equation (2) and putting $\sqrt{2c'/k_2} = c$, we get

$$x = c\sqrt{n} \qquad (4)$$

as the relation between critical frequency and mean concentration of photoproducts in the reversible photochemical system. This value of x when put into equation (3) of the steady state gives

$$\frac{KI}{2c} = \frac{n}{a/c - \sqrt{n}} \qquad (5)$$

which should describe the dependence of critical frequency on intensity,

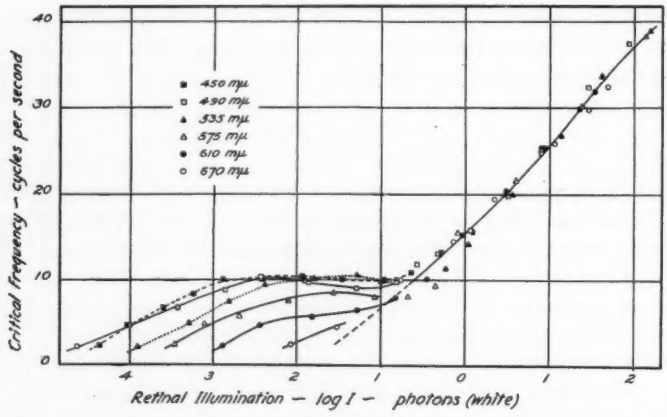


FIGURE 6

Relation of critical frequency to log I for a 5° off-center retinal position for different parts of the spectrum. The upper portions of the data have been made coincident with the white data under similar conditions.

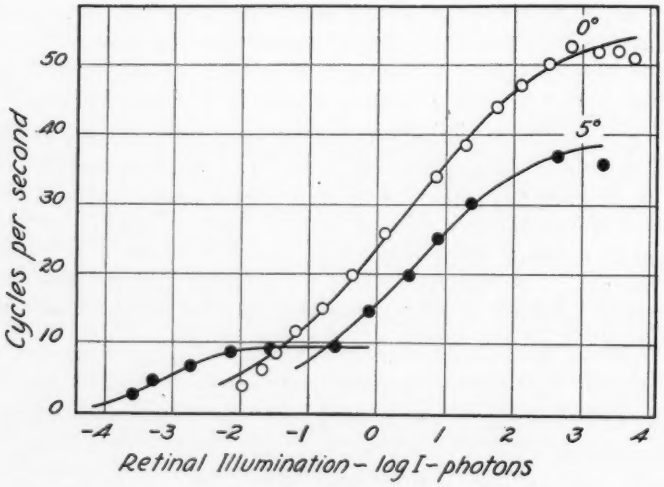


FIGURE 7

Theoretical representation of data for S. H. given in figure 4. The curve through the low intensity section of the 5° off-center data is drawn from equation (5) whereas the other two curves are drawn from equation (6). The specific values of the constants are given in the text.

in a single homogeneous system such as a rod or a cone. In equation (5) a is 100 per cent, while K and c are constants to be found from the data.

Figures 4 and 5 show the relation of equation (5) to the data of S. H. and C. D. V. taken from figures 2 and 3, respectively. For S. H. the line through the 5° off-center rod data is given by $20{,}600I = n/(2.95 - \sqrt{n})$. For C. D. V. the line through the rod data of figure 5 has the equation $7720I = n/(3.08 - \sqrt{n})$. The adequacy of equation (5) as a description of the rod measurements shows that the flicker function of the rods behaves as if it were controlled by the initial photochemical process in photoreception.

For the flicker function of the cones equation (5) yields a curve whose

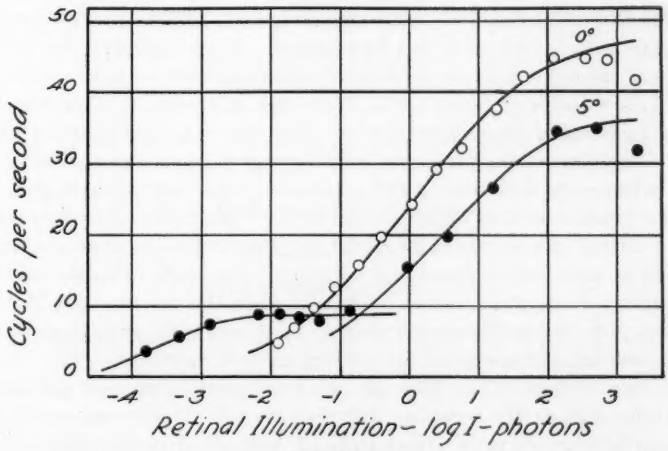


FIGURE 8

Theoretical representation of data for C. D. V. from figure 3. The curve through the low intensity section of the 5° off-center data is drawn from equation (5); the other two curves are drawn from equation (6). The specific values of the constants are given in the text.

slope is too great by a factor of about 2. To correct for this we have arbitrarily attached an exponent α to the intensity I. The value of α is very nearly 0.50, which means that the intensity factor enters as its square root. This is not uncommon in photochemical experience.[14] Equation (5) then becomes

$$\frac{KI^{\alpha}}{2c} = \frac{n}{a/c - \sqrt{n}} \qquad (6)$$

which may now be used to compare with the data.

For S. H. in figure 4, the curve through the cone portion of the 5° off-center data is given by $6.93I^{0.55} = n/(6.08 - \sqrt{n})$, and for C. D. V.

the corresponding line in figure 5 is given by $6.74I^{0.50} = n/(6.33 - \sqrt{n})$. For the foveal data of S. H. in figure 4, the curve is $11.72I^{0.52} = n/(6.93 - \sqrt{n})$, and for the corresponding data in figure 5 for C. D. V. the curve is $9.15I^{0.44} = n/(7.48 - \sqrt{n})$.

These numerical comparisons between the measurements and equations (5) and (6) show that the initial step in photoreception considered as a reversible photochemical reaction is able to furnish a first order description of the relation between the critical frequency of flicker and the intensity. It has already been shown[13] that Talbot's law follows from the same considerations. It is therefore of interest to point out that the behavior of the critical frequency during light and dark adaptation as well as the rise and fall of critical frequency with increasing ratio of light to dark periods also may be derived from such a formulation. These aspects of the flicker problem, the full details of the present measurements, and a consideration of current theories of flicker are all dealt with in a paper to appear in the *Journal of General Physiology*.

7. *Summary.*—Measurements of the critical fusion frequency for white light, when made with the rod-free area of the fovea yield a single, sigmoid relation between critical frequency and log I. Similar measurements made with a retinal area 5° off-center containing rods and cones show a relation possessing two clearly separated sections. The high intensity section resembles the one obtained for the fovea, while the low intensity section is new. The two sections are separated by a horizontal limb at about 10 cycles per second maintained for 1.25 log units of intensity.

The two sections of the data are separate functions of the rods at low intensities and of the cones at high intensities. This is borne out by measurements made with retinal areas 15° and 20° from the fovea where the ratio of rods to cones is anatomically greater than at 5° off-center. As a result the two sections of the data become still further separated than at 5° off-center.

Further confirmation of the rod-cone character of the peripheral data is furnished by measurements of flicker for 5° off-center, with different portions of the visible spectrum. Following the established variations in relative sensibility of the rods and cones at different wave-lengths, the two sections of the data show a corresponding separation along the intensity axis for different wave-lengths. The separation is greatest in the violet and least in the red.

A theoretical treatment of the measurements in terms of a reversible photochemical system shows that a first order quantitative description of the data can be given by the properties of such a system considered as the initial event in photoreception.

* Fellow (1929–30) of the Donders Foundation (Holland).

[1] Ferry, E. S., *Am. J. Sci.*, **44** (3), 192 (1892).

[2] Porter, T. C., *Proc. Roy. Soc.*, **70**, 313 (1902).
[3] Ives, H. E., *Phil. Mag.*, **24** (6), 352 (1912).
[4] Allen, F., *J. Opt. Soc. Amer.*, **13**, 383 (1926).
[5] Lythgoe, R. J., and Tansley, K., *Med. Res. Counc., Spec. Rep. Ser.*, No. 134, London (1929).
[6] Granit, R., and Harper, P., *Am. J. Physiol.*, **95**, 211 (1930).
[7] Allen, F., *Phil. Mag.*, **38**, 81 (1919).
[8] Dieter, W., *Arch. f. Ophthal.*, **113**, 141 (1924).
[9] Hyde, E. P., Forsythe, W. E., and Cady, F. E., *Astrophys. J.*, **48**, 65 (1918).
[10] Hecht, S., and Williams, R. E., *J. Gen. Physiol.*, **5**, 1 (1922).
[11] Adrian, E. D., and Matthews, R., *J. Physiol.*, **64**, 279 (1927).
[12] Hecht, S., *Ergebn. d. Physiol.*, **31**, 243 (1931).
[13] Hecht, S., and Wolf, E., *J. Gen. Physiol.*, **15**, 369 (1932).
[14] Griffith, R. O., and McKeown, A., *Photo-Processes in Gaseous and Liquid Systems*, London (1929).

COLOR CHANGES IN FUNDULUS WITH SPECIAL REFERENCE TO THE COLOR CHANGES OF THE IRIDOSOMES

BY KENDALL W. FOSTER

ZOÖLOGICAL LABORATORIES, HARVARD UNIVERSITY

Communicated March 22, 1933

As suggested by Connelly (1925), the iridocytes (Pouchet, 1876; guanophores, Ballowitz, 1912) play an inactive rôle in the color changes incited in the fish *Fundulus heteroclitus L.* by the colors of the backgrounds over which they are placed. For example, the yellow chromatophores (xanthophores) when partly "expanded"[1] cover the blue iridocytes which comprise the stratum argenteum (Cunningham and MacMunn, 1893; l'argenture of Pouchet, 1876; reflecting layer of Connelly, 1925). The blue color which emanates from the guanin crystals of these iridocytes viewed together with the translucent yellow of the xanthophores gives the green color which is characteristic of fish that have been kept upon a green background. Likewise, in fish kept over blue backgrounds, the xanthophores are maximally "contracted"[1] (Fries, 1931), revealing the blue of the stratum argenteum. The black cells (melanophores) are partly "expanded," shading the stratum argenteum beneath, and darkening the shade, but not greatly obscuring the blue color.

Certain iridosomes (Cunningham and MacMunn, 1893; Ballowitz, 1912), i.e., isolated groups of iridocytes, in Fundulus are associated with large melanophores, located in the deeper portions of the dermis but external to the stratum argenteum. These iridosomes form "chromatophore combinations," called "melaniridosomes" in other species of fish by Ballowitz (1912), and by Becher (1929). These iridosomes, together

with the reflecting patches (Connelly, 1925) composed of iridocytes found in male Fundulus, play a passive though not as important a rôle in the adaptive color changes of the fish as do the iridocytes of the argenteum. These iridosomes and iridocytes, composed of guanin crystals, are always blue or green in color, regardless of the color of the background upon which the fish has sojourned, except under certain conditions to be described below.

No account of changes in color of iridosomes, independent of the effects of chromatophores mentioned above, has been found in the literature. It is purposed here to report briefly certain interesting color changes of the iridosomes of Fundulus. These activities have developed in connection with a problem suggested by Prof. G. H. Parker.

When stimulated by direct sunlight, or the light from a Silverman Illuminator lamp, the iridosomes and the reflecting patches change color very rapidly. They appear blue or green in color when first exposed to the light, and within approximately five seconds change to an orange or wine red color. After this change, within forty seconds to two minutes of the time they were stimulated, either in the dark or in light, the iridosomes regain their original color. After responding to a stimulus the iridosomes remain refractory to further stimulation for approximately twenty minutes. If the intensity of the Silverman Illuminator light is reduced, so that the color of the iridosomes is just visible with the microscope, the change in color progresses more slowly. When the light is first thrown on, the general character of the color of the iridosome is green or blue; while certain regions in it, one or two plate-like guanin crystals, are light green or yellowish in color. The color of the iridosomes in general changes through yellowish green, to yellow, to orange or wine red. At the end of such a color change the portion of the iridosome which was originally blue or green becomes orange, while the few crystals which were originally yellow or light green become wine red in color. Rarely is an iridosome found showing only one color. Generally they show a mixture with one color predominating, as in the case described. The recovery phase which follows the initial color change progresses in the reverse order, from the red toward the blue end of the spectrum. These color changes on the part of the iridosomes occur in greater or less clearness, depending upon the state of "expansion" or "contraction" of the melanophores and xanthophores. They show most clearly in fish which have been upon a blue background and which are blue in color. The melanophores are then partly "expanded" and the xanthophores are maximally "contracted" (Fries, 1931). The colors of the iridosomes are darker in shade because of the melanophore just behind them. The iridosome colors are not conditioned in their appearance by the overlying layer of xanthophores; for when expanded, the xanthophores are a very light translucent yellow, and cause the iridosomes

below them, which are therefore viewed more or less through the xanthophores, to appear more green or yellow than they really are. This appearance is probably due on the one hand to blue light from the iridosome, which, passing between processes of the expanded xanthophores, mingles with the yellow from the xanthophores to give a green color. On the other hand, those xanthophores which cover an iridosome in the dispersed condition of their pigment probably do not absorb all the blue light from the iridosome, but transmit some of it. This transmitted light, mixed with yellow light from the xanthophores, presents a green, or even a yellowish green, the exact color depending upon the amount of the transmitted blue light.

The fish studied in Cambridge showed less reactiveness on the part of their iridosomes than those studied at the United States Bureau of Fisheries Laboratory at Woods Hole. This may be due to the less favorable condition of the material used in Cambridge, or to the fact that the fish studied at Woods Hole were observed through the summer months, during and after the breeding season. It has been noted by others (Parker, 1930), that the color changes of fish called forth during the breeding season by the color of the background are much more pronounced than during the rest of the year. The humoral state of the body is undoubtedly very different during the breeding season, and this may place the iridosomes in a particularly reactive condition.

The iridocytes of the stratum argenteum have not been observed to change color, and only those of the melaniridosomes and the reflecting patches, which are more superficially located in the skin, show this phenomenon.

Under the high magnifications possible with the Ultropak microscope, the individual crystals may be more or less clearly seen in certain iridosomes of the living fish. During the color change of the iridosomes no movement of the crystals was observed. The crystals appear to be long thin plates lying more or less flat on a plane parallel with the surface of the skin. Occasionally a crystal is seen turned up on edge, and faint striations are noted. This seems to indicate either that the crystals are laminated or that they adhere with their broad surfaces together. According to Pfund (1917) and Bancroft (1919) mother-of-pearl in mollusk shells owes its colors to two causes: first, to the diffraction of light by the grating-like structure of the terminal edges of alternating laminae of aragonite ($CaCO_3$) and conchiolin, which overlap one another at the surface like slates on a roof; and secondly, to the interference of light reflected from parallel laminae of equal thickness. The colors that emanate from the guanin crystals are probably not caused by diffraction, since not much evidence of a grating is observed in the crystals. The interference of light reflected from parallel laminae of the crystals, if such laminae exist, or

perhaps from the thin plate-like crystals themselves, seems the best explanation of these colors. If the crystals are laminated, they must nevertheless be composed of only one substance, for the more recent investigators (Taylor, 1925) are of the opinion that the guanin in such crystals is pure and not combined with other substances.

The iridosomes will respond locally to mechanical or electrical stimulation. They do not respond to heat, except that at higher temperatures than room temperature they respond more generally and completely upon stimulation by light. Heat may then be regarded as having an accelerating effect upon the color change.

The environmental medium about the iridosomes has a profound effect upon their color and their reactiveness when stimulated. If two or three scales are removed from a fish, thus breaking the epidermis and subjecting the iridosomes to the osmotic and ionic effects of the environmental medium, the following effects are noted. If such a fish is placed in tap water, in five minutes the iridosomes of the scale-free region are yellow in color rather than blue or green. If the same fish is placed in sea water, in five minutes the same iridosomes become dark blue or green. This reaction may be reversed several times with the same iridosomes. In such fish, in which only a relatively small area of the dermis is exposed to the environmental medium, the responses of the iridosomes to light stimulation are but slightly inhibited in tap water; but greatly inhibited in sea water. Isolated pieces of scale-free skin placed in tap and later in sea water show similar reactions; but the color changes of scale-free skin in response to light are inhibited. It is evident from these results that the environment may condition the color of the iridosomes, and also that it may inhibit their response to light in proportion to the degree of exposure.

The question arises as to whether these reactions of the iridosomes are under the control of the nervous system. If light is allowed to stimulate a very small region of the skin under observation, the iridosomes respond in this region by changing color. When a neighboring region of the skin of the same fish is illuminated in the same fashion, the iridosomes are found not to have changed color in response to the illumination of the first region, but respond only to the second stimulation. This indicates that there is no conduction to the neighboring iridosomes of the excitation which initiated the response of the first group of iridosomes, but does not remove the possibility of a local reflex arc in which the receptor is located in or near the effector. To test for this possibility, the body wall was cut in such ways, on several fish, that the sum of the cuts would have isolated a piece of body wall from the fish, provided all the cuts had been made in one fish. In the center of this piece of body wall the iridosomes responded to the stimulation of light regardless of which side of this point the body wall was cut. This experiment proves that the iridosomes in their re-

sponses to stimulation are independent of the nervous system. The local and extremely rapid nature of the response would also seem to remove the possibility of the intervention of any humoral controlling factor in these reactions.

It may be concluded that the iridosomes of Fundulus respond directly to certain definite stimuli. They show fairly definite reaction, recovery and refractory phases in their response and are independent of the nervous system in their reactions. Their color changes progress from the colors at the blue end of the spectrum through the intervening colors, in the order of increasing wave-lengths, to colors at the red end of the spectrum. In the recovery phase, following almost immediately upon the reaction phase, the colors go down the spectrum in the order of decreasing wave-lengths, finally reaching a color near the blue end. This indicates a progressive change in the reaction of the iridosomes to stimulation, and also in their recovery a reverse process. This in turn suggests that possibly the laminae of the crystals, if they exist, or the crystals themselves, may in some way be thickened progressively during the reaction phase and thinned progressively during the recovery phase. The iridosomes appear to be active structures in Fundulus, though they are not active in producing the color responses of the animal to the color of the background upon which it is placed. In this latter connection the iridosomes play an important but only a passive rôle. The adaptive color changes in Fundulus skin are due entirely to the activity of the melanophores and xanthophores in covering or uncovering the guanin deposits by their processes of "expansion" or "contraction."

[1] The terms "expansion" and "contraction" as applied to the activities of chromatophores are used for reasons of convenience and precedence and should not be construed as implying amoeboid activity on the part of pigment cells.

LITERATURE CITED

Ballowitz, E., "Über chromatische Organe in der Haut von Knochenfischen," *Anat. Anz.*, **42**, 186–190 (1912).

Bancroft, W. D., "The Colors of Colloids. V. Metallic and Vitreous Luster," *Jour. Phys. Chem.*, **23**, 289–347 (1919).

Becher, H., "Über die Verwendung des Opak-Illuminators zu biologischen Untersuchungen nebst Beobachtungen an den lebenden Chromatophoren der Fischhaut in auffallenden Licht," *Zeit. wiss. Mik.*, **46**, 89–124 (1929).

Connelly, C. J., "Adaptive Changes in Shades and Color of *Fundulus*," *Biol. Bull.*, **48**, 56–77 (1925).

Cunningham, J. T., and MacMunn, C. A., "On the Coloration of the Skins of Fishes, Especially of Pleuronectidae," *Phil. Trans. Roy. Soc. London*, **B184**, 765–812 (1893).

Fries, E. F. B., "Color Changes in Fundulus, with Special Consideration of the Xanthophores," *Jour. Exp. Zoöl.*, **60**, 389–426 (1931).

Parker, G. H., "Chromatophores," *Biol. Rev.*, **5**, 59–90 (1930).

Pfund, A. H., "The Colors of Mother-of-Pearl," *Journ. Franklin Inst.*, **184**, 453–464 (1917).

Pouchet, G., "Des changements de coloration sous l'enfluence des nerfs," *Journ. Anat. Physiol.*, **12**, 1–90, 113–165 (1876).

Taylor, H. F., "Pearl Essence: Its History, Chemistry and Technology," *Rep. Comm. Fish.*, Washington, **1925**, 15–36 (1926).

THE CAUCHY-GOURSAT THEOREM FOR RECTIFIABLE JORDAN CURVES

By J. L. Walsh

Department of Mathematics, Harvard University

Communicated March 17, 1933

In a recent paper[1] Kamke stated that the property expressed by Cauchy's integral theorem had never been proved for the case of a function analytic interior to an arbitrary rectifiable Jordan curve, continuous in the corresponding closed region. A proof was then supplied by Denjoy.[2] The following proof is much more immediate than that of Denjoy, although not so elementary.

THEOREM I. *If C is a rectifiable Jordan curve and if the function $f(z)$ is analytic interior to C, continuous in the corresponding closed region, then we have*

$$\int_C f(z)\,dz = 0.$$

The integral of an arbitrary polynomial $p(z)$ over C is zero, for that integral can be expressed as the limit as n becomes infinite of the integral of $p(z)$ over a suitably chosen closed polygon π_n whose vertices lie on C; the latter integral is clearly zero. The function $f(z)$ of Theorem I, being analytic interior to C and continuous on and within C, can be represented *in the closed interior of C* as the limit of a uniformly convergent sequence of polynomials.[3] This sequence can be integrated over C term by term, so Theorem I is established.

Theorem I extends easily to the case of a limited region D bounded by a finite number of non-intersecting rectifiable Jordan curves, if $f(z)$ is analytic interior to D, continuous in the corresponding closed region. In such a closed region the function $f(z)$ can be expressed as the limit of a uniformly convergent sequence of rational functions of z whose poles lie exterior to the closed region.[4] The integral of such a rational function over the complete boundary of D is zero; hence the corresponding integral of $f(z)$ is also zero.

In particular, Cauchy's integral *formula* is valid under the hypothesis of Theorem I, or under the more general hypothesis just mentioned.

[1] *Math. Zeit.*, **35**, 539–543 (1932).
[2] *Paris Compt. Rend.*, **196**, 29–33 (1933).
[3] Walsh, *Math. Annal.*, **96**, 430–436 (1926).
[4] Walsh, *Ibid.*, 437–450 (1926).

AN INEQUALITY FOR LEGENDRE SERIES COEFFICIENTS

BY M. F. ROSSKOPF

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY

Communicated March 28, 1933

Let the set $\{\varphi_n(x)\}$ be an ortho-normal set of functions on the interval (a, b) and let M be a constant such that

$$|\varphi_n(x)| \leqq M, \quad n = 0, 1, 2, \ldots;$$

then the Fourier expansion of any integrable function $f(x)$ will be

$$f(x) \sim \sum_0^\infty c_n\varphi_n(x),$$

where

$$c_n = \int_a^b f(t)\varphi_n(t)dt.$$

We introduce the notation

$$J_p(f) = J_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}, \quad S_{p'}(f) = S_{p'} = \sum_0^\infty |c_n|^{p'},$$

where $1 < p \leqq 2$, $p' \geqq 2$, $\frac{1}{p} + \frac{1}{p'} = 1$. It will be assumed throughout that p and p' satisfy these relations.

In terms of this notation F. Riesz's[1] theorems can be stated, for a uniformly bounded ortho-normal set of functions, as follows.

(A) If $f(x)cL_p$, then

$$S_{p'} \leqq M^{\frac{2-p}{p}} J_p.$$

(B) If the series $\sum_0^\infty |c_n|^p$ is convergent, then the constants c_n are the Fourier coefficients of a function $f(x)cL_{p'}$, and, moreover,

$$J_{p'} \leqq M^{\frac{2-p}{p}} S_p.$$

As was called to my attention by Professors Hille and Tamarkin, in the case of the expansion of the function

$$f(x) = \left(\frac{2}{1 \quad x}\right)^\alpha$$

in normalized Legendre polynomials F. Riesz's theorems do not hold. Stieltjes[2] considered this function and showed that for the convergence of its Legendre series, besides assuming $-1 < x < +1$, it is necessary to take $\alpha < \frac{3}{4}$; from the asymptotic value of the coefficients this is easily seen, since

$$c_n = \frac{2^{1/2}(2n+1)^{1/2}\Gamma(\alpha+n)\,\Gamma(1-\alpha)}{\Gamma(\alpha)\Gamma(n-\alpha+2)} \sim \frac{2\Gamma(1-\alpha)}{\Gamma(\alpha)}\, n^{2\alpha-3/2}.$$

The above function belongs to L_p for every $p < \frac{1}{\alpha}$, whereas the series $\sum_0^\infty |c_n|^{p'}$ diverges whenever $\alpha \geqq \frac{3}{4}$; thus it is seen that Riesz's first theorem does not apply to Legendre series.

The problem now is to modify J_p into J_p^0 so that the Legendre coefficients of a certain class of functions would satisfy a new inequality. In particular it is desirable to obtain an inequality which would take care of this function of Stieltjes; i.e., such that $J_p^0(f)$ and $S_{p'}(f)$ would converge and diverge together. The theorems below are of exactly that nature.

Consider the set of normalized Legendre polynomials defined on the interval $(-1, +1)$

$$\left\{\left(\frac{2n+1}{2}\right)^{1/2} P_n(x)\right\}$$

and the formal expansion of an arbitrary integrable function $f(x)$ in its Legendre series

$$f(x) \sim \sum_0^\infty \left(\frac{2n+1}{2}\right)^{1/2} c_n P_n(x), \quad c_n = \left(\frac{2n+1}{2}\right)^{1/2} \int_{-1}^{+1} f(t) P_n(t)\,dt.$$

We introduce the new definitions,

$$J_p^0(f) = J_p^0 = \left(\int_{-1}^{+1} \frac{|f(t)|^p}{(1-t^2)^{1/2-p/4}}\,dt\right)^{1/p}, \quad S_{p'}(f) = S_{p'} = \left(\sum_0^\infty |c_n|^{p'}\right)^{1/p'},$$

where p and p' satisfy the same relations as before. Then the following theorems can be stated.

THEOREM 1. If $f(x)(1-x^2)^{1/4-1/2p} \, cL_p$, then

$$S_{p'} \leqq J_{p'}^0.$$

THEOREM 2. If the series $\sum_0^\infty |c_n|^p$ converges, then the constants c_n are

the Legendre coefficients of a function $f(x)$ which is such that $f(x)(1 - x^2)^{1/4 - 1/2p'} cL_{p'}$, and, moreover,

$$J^0_{p'} \leqq S_p.$$

The methods used to prove these theorems are essentially those of M. Riesz.[3] The whole problem reduces to the definition of a linear functional transformation $T(f)$ with modulus $M^*_{\alpha,\gamma} = M^*_{1/p,\,1/p'}$. After $M^*_{1,0}$ and $M^*_{1/2,\,1/2}$ are calculated, the convexity of $\log M^*_{\alpha,\gamma}$ is assured by M. Riesz's theorems; from these it also follows that one can interpolate on the line segment determined by the end points (1, 0) and ($1/2$, $1/2$).

Analogous methods can be applied to a wide class of ortho-normal functions which are not uniformly bounded; hence to which F. Riesz's theorems in their initial form cannot be applied. A detailed discussion of all these problems will be published elsewhere.

[1] F. Riesz, *Math. Zeit.*, **18**, 117–124 (1923). For the case of trigonometric series see F. Hausdorff, *Math. Zeit.*, **16**, 163–169 (1923). In this paper is also given a list of W. H. Young's papers on the subject.

[2] *Correspondance d'Hermite et Stieltjes*, Paris (1905). Gauthier-Villars, **2**, letter 249, page 46.

[3] M. Riesz, *Acta Math.*, **49**, 465–497 (1926). In particular see Theorems V and VI.

THE GEOMETRY OF DEGENERATE HEAT FAMILIES

BY AARON FIALKOW

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY

Communicated March 22, 1933

1. In this paper, we wish to consider a special phase of the geometry of plane heat families.

Professor Kasner has defined a *plane heat family* or *family of heat curves* as the complete set of isothermals for all instants of time, obtained when heat flows by conduction in the plane.[1] The mathematical equivalent of this definition of heat curves proceeds without difficulty. The temperature, v, is a function of the position in the plane, x, y, and the time, t, say, $\varphi(x, y, t)$. Then the equation

$$v = \varphi(x, y, t) \tag{1}$$

defines the family of heat curves if we think of v and t as parameters, where the function $\varphi(x, y, t)$ may be any solution of Fourier's heat equation (for planar flow)

$$\Delta\varphi = \varphi_t \tag{2}$$

where $\Delta\varphi = \varphi_{xx} + \varphi_{yy}$.

In general, heat curves form a two-parameter family, the parameters being the temperature and the time; in certain special cases, the family may involve only one essential parameter. This latter situation would clearly be realized if the heat equation (1) could be written in the form

$$g(x, y, \theta(v, t)) = 0. \tag{3}$$

Approaching the question from the standpoint of differential equations, we obtain another method for describing these circumstances. In general, the elimination of both parameters would yield a differential equation of the second order. Where there is one essential parameter, the elimination could be accomplished by a single differentiation, and hence the resulting differential equation would be of the first order. A set of heat curves which involves only one essential parameter has been called a *degenerate heat family* by Kasner.

Kasner has shown that there are three distinct types of degenerate heat families. The temperature function for these types takes the special forms:

$$v = u(x, y).$$

This is the case of a steady flow of heat. The identical isothermals appear independently of the time and $u(x, y)$ satisfies Laplace's equation $\Delta u = 0$.

$$v = u(x, y) + t.$$

The temperature is linear in the time and $\mu(x, y)$ is a solution of a special form of Poisson's equation $\Delta u = 1$.

$$v = u(x, y) \cdot e^{t}$$
$$v = u(x, y) \cdot e^{-t}.$$

The temperature is an exponential function of the time and $u(x, y)$ obeys the corresponding equation of Helmholtz $\Delta u = u$ or $\Delta u = -u$.

To these should be added families of parallel lines and of concentric circles. These come under all three types above, except that the temperature may appear in a more complicated form.

In a study of the geometry of degenerate heat curves, we would therefore be led to a consideration of the geometry of the single infinitude of plane curves

$$u(x, y) = c \tag{4}$$

where $u(x, y)$ is a solution of one of the four equations

$$\Delta u = 0 \tag{I}$$
$$\Delta u = 1 \tag{II}$$
$$\Delta u = u \tag{III$_1$}$$
$$\Delta u = -u. \tag{III$_2$}$$

This is Kasner's main result—that all families of degenerate heat curves are defined by the equations of Laplace, Poisson or Helmholtz. All the solutions of these equations are included among the solutions of the more general equation

$$\Delta u = f(u). \tag{IV}$$

By the geometry of a one-parameter family, we understand, the geometric properties of the associated orthogonal net consisting of (4) and

$$u_x y' - u_y = 0. \tag{5}$$

At an arbitrary point of the plane, the curvature of (4) and its tangential and normal derivatives are represented by γ, γ_s and γ_n. The corresponding symbols for (5) are Γ, Γ_S and Γ_N. An analogous notation is used for higher derivatives. The geometric properties of the orthogonal net are relations among γ, Γ and their derivatives. Geometric properties of this sort would be visible in a photograph of the orthogonal net. In this paper, we answer the question: What are characteristic geometric properties of degenerate heat families?[2] In the course of our work, we derive several theorems not directly connected with heat families.

2. Certain geometric relations are identities; that is, they are properties of all orthogonal nets. Geometric identities are given by

THEOREM I. *Every orthogonal net has one geometric property of the first order*

$$\gamma_n + \Gamma_N - (\gamma^2 + \Gamma^2) = 0$$

and two geometric properties of the second order

$$\Gamma\gamma_n + \gamma\gamma_s - \gamma_{ns} + \gamma_{sn} = 0$$
$$\gamma\Gamma_N - \Gamma\Gamma_S + \Gamma_{NS} - \Gamma_{SN} = 0.$$

The first order property is *Lamé's Relation*.[3] It is the simplest geometric identity. The two higher order properties are apparently new results.

What can be said of the geometry of the orthogonal net when $u(x, y)$ is a solution of (IV)? The discussion of this situation divides into two cases, according as the family is non-parallel or parallel. Our results are:

THEOREM II. *The necessary and sufficient condition that a one-parameter, non-parallel family of plane curves, $u(x, y) = c$, be a solution of $\Delta u = f(u)$ is that*

$$\Gamma(\gamma_{ss} - \Gamma_{NS}) + (\Gamma_N - \Gamma^2)(\gamma_s + \Gamma_S) = 0.$$

THEOREM III. *If the family of plane curves, $u(x, y) = c$, is parallel and is also a solution of $\Delta u = f(u)$, it must be a family of parallel straight lines or concentric circles. Conversely, every family of parallel straight lines or concentric circles can be parametered so that it is a solution of $\Delta u = f(u)$ for every $f(u)$.*

By appropriate specialization of the form of $f(u)$, theorems are obtained which give a complete geometric classification of degenerate heat families. A partial geometric classification is known. A characteristic property[4] has been given for the Laplace type, namely, $\gamma_s + \Gamma_S = 0$, and, in his Seminar of 1917, Kasner has established that the other two types each possess two fourth order properties.[1] For the rest of the problem, namely, discovering these two properties explicitly and furthermore, proving that they are sufficient to characterize these types, we open and close the discussion. The complete results are:

THEOREM IV. *The necessary and sufficient condition that a one-parameter, non-parallel family of plane curves be a degenerate heat family is that it be a solution of the geometric equation*

$$\Gamma(\gamma_{ss} - \Gamma_{NS}) + (\Gamma_N - \Gamma^2)(\gamma_s + \Gamma_S) = 0$$

and either

$$\gamma_s + \Gamma_S = 0$$

or $$2\Gamma(\Gamma_{SS} + \gamma_{ns}) + (2\gamma\Gamma - \gamma_s - 3\Gamma_S)(\gamma_s + \Gamma_S) = 0$$

or $$2\Gamma(\Gamma_{SS} + \gamma_{ns} + 2\Gamma) + (2\gamma\Gamma - \gamma_s - 3\Gamma_S)(\gamma_s + \Gamma_S) = 0$$

or $$2\Gamma(\Gamma_{SS} + \gamma_{ns} - 2\Gamma) + (2\gamma\Gamma - \gamma_s - 3\Gamma_S)(\gamma_s + \Gamma_S) = 0.$$

In these four cases, the degenerate heat family is respectively of Type (I), (II), (III_1) *or* (III_2).

Thus we can characterize completely, not only the equation of Laplace, but also those of Poisson and Helmholtz. Of course, if $\gamma_s + \Gamma_S = 0$, the first equation given above becomes identically zero.

Kasner has shown that the only degenerate heat families which appear in more than one type are parallel straight lines and concentric circles. Each such family may be considered a degenerate heat family of Types (I), (II) and (III).[1]

3. Since all quantities in this discussion are invariant with respect to the group of plane motions, we may specialize the arbitrary element (x, y, y') to $(0, 0, 0)$ without loss of generality. For this element, the expressions for γ, Γ and their derivatives are greatly simplified. In our work, we require the formulae for γ, Γ and all their derivatives of the first and second order. Fourteen equations are actually obtained since geometric differentiation is not commutative in general. These equations involve the partial derivatives of $u(x, y)$ up to the fourth order. By proper manipulation these equations may be made linear in the partial derivatives of $u(x, y)$. We thus obtain a system of fourteen linear equations in twelve unknowns. The coefficient matrix is of rank six. The solution of certain six of the equations gives us expressions for the fourth and third order partial derivatives of $u(x, y)$ in terms of γ, Γ and their derivatives and u_y, u_{yy} and u_{yyy}. The substitution of these values in the remaining equa-

tions yields three relations among γ, Γ and their derivatives which are stated in Theorem I. The third of these relations may be derived from the second by interchanging γ, Γ; s, $-N$; and n, S.

4. If we assume that $u(x, y)$ is a solution of (IV), four conditions on $u(x, y)$ may be obtained by successive partial differentiation with respect to x and y. Since Δu and $f(u)$ are invariant under the group of plane motions, we may assume $u_x = 0$, and substitute for the partial derivatives in these equations the values found for the element (0, 0, 0). If we eliminate the partial derivatives with respect to y only from the resulting equations and make use of Theorem I, we obtain

$$\Gamma(\gamma_{ss} - \Gamma_{NS}) + (\Gamma_N - \Gamma^2)(\gamma_s + \Gamma_S) = 0 \tag{6}$$

$$2\Gamma(\Gamma_{SS} + \gamma_{ns} + 2\Gamma f'(u)) + (2\gamma\Gamma - \gamma_s - 3\Gamma_S)(\gamma_s + \Gamma_S) = 0. \tag{7}$$

Thus (6) is a necessary geometric property of (IV) and hence, in particular, of (I), (II) and (III). Equation (7) is not, in general, a purely geometric property but depends on $u(x, y)$. Since, however, for (II) and for both forms of (III), $f'(u) = 0, +1$ or -1, a special form of (7) is an additional property of each of these types.

To prove that (6) is also sufficient to characterize (IV), it will suffice to show that, on the basis of (6), an $F(u)$ can be found such that

$$\Delta F(u) = F''(u) \cdot \Delta_1 + F'(u) \cdot \Delta_2 = f(u) \tag{8}$$

where Δ_1 is the *first differential parameter*, $u_x^2 + u_y^2$, and Δ_2 $(= \Delta u)$ is the *second differential parameter*, $u_{xx} + u_{yy}$. This is a consequence of the fact that $F(u) = c_1$ is the same set of curves as (4). We note the following formulae which can easily be verified.

$$\Gamma = -\frac{(\Delta_1)_s}{2\Delta_1} \tag{9}$$

$$\gamma_s + \Gamma_S = -\Delta_1^{1/2}\left(\frac{\Delta_2}{\Delta_1}\right)_s. \tag{10}$$

Since $s = -N$, we may write (6) in the form

$$\Gamma(\gamma_s + \Gamma_S)_s - \Gamma_S(\gamma_s + \Gamma_S) - \Gamma^2(\gamma_s + \Gamma_S) = 0.$$

Substituting the values given in (9) and (10) we obtain

$$2\Delta_1^2[(\Delta_1)_s(\Delta_2)_{ss} - (\Delta_1)_{ss}(\Delta_2)_s] = 0.$$

If $\Gamma \not\equiv 0$, the orthogonal trajectories are not all straight lines; that is, the family is not parallel. In this case we may divide by $2\Delta_1^2(\Delta_1)_s^2$. Hence

$$\left[\frac{(\Delta_2)_s}{(\Delta_1)_s}\right]_s = 0. \tag{11}$$

Since $\frac{du}{ds} = 0$, on integrating this equation, we have

$$\frac{(\Delta_2)_s}{(\Delta_1)_s} = \Phi(u). \tag{12}$$

We can now determine an $F(u)$ such that

$$\frac{F''(u)}{F'(u)} = -\Phi(u) = -\frac{(\Delta_2)_s}{(\Delta_1)_s}. \tag{13}$$

This $F(u)$ is a solution of (8) as may be seen by integrating (13). This completes the proof of Theorem II.[5]

If $\Gamma \equiv 0$, the orthogonal trajectories are all straight lines and the given family is, of course, parallel. Then (6) is satisfied. This means that, besides the solutions of (6) given in Theorem II, every parallel family is also a solution of (6). From (9), Δ_1 is a function of $u(x, y)$. If we assume a solution of (8) exists, Δ_2 must be a function of $u(x, y)$. Therefore $\left(\frac{\Delta_2}{\Delta_1}\right)_s = 0$. Referring to (10), $\gamma_s = 0$, that is, γ is constant along each curve. This means that the curves are parallel straight lines or concentric circles. Conversely, if Δ_2 is a function of $u(x, y)$, it is clear that an $F(u)$ may be found to satisfy (8) for an arbitrary choice of $f(u)$. This proves Theorem III.

5. The theorem relating directly to degenerate heat families follows easily. The statement for the Laplace type in Theorem IV is simply the known result. The remainder of this theorem is proved by noting that the first equation states that $u(x, y)$ satisfies (IV) and, since $f'(u)$ is invariant under the group of plane motions, by comparison with (7), the last three equations lead to the conclusion that $\Delta u = a_1$ or $u + a_2$ or $-u + a_3$. For each such type of $u(x, y)$, it is always possible to choose an $F(u)$ such that $\Delta F(u) = 1$ or $F(u)$ or $-F(u)$.

[1] See Kasner, "Geometry of the Heat Equation: First Paper," *Proc. Nat. Acad. Sci.*, **18**, 475 (June 1932); "Second Paper," *Ibid.*, **19**, 257 (Feb. 1933); and abstract in *Bull. Amer. Math. Soc.*, **23**, 302 (1917).

[2] See abstract in *Bull. Amer. Math. Soc.*, 803 (Nov. 1932).

[3] See Lamé, "Mémoire Sur Les Coordonnées Curvilignes," *Jour. Mathémat.*, **5**, 347 (1840).

[4] Other methods of characterizing the Laplace type (that is, ordinary isothermal families) have been given by Kasner, *Math. Annal.*, **59**, 352 (1904), and *Princeton Colloquium Lectures*, Differential-geometric aspects of dynamics.

[5] A shorter proof of the first part of Theorem II can be given by reversing the argument in the sufficiency proof. However, this method does not yield the results of Theorem I.

ON STATISTICAL METHODS IN THE THEORY OF ALMOST-PERIODIC FUNCTIONS

BY E. K. HAVILAND

DEPARTMENT OF MATHEMATICS, THE JOHNS HOPKINS UNIVERSITY

Communicated April 14, 1933

The purpose of the present paper is to show that every almost-periodic complex function[1] of a real variable possesses a distribution function. The corresponding theorem for real almost-periodic functions has been proved by Wintner, first by the use of infinite matrices[2] and later[3] by the Helly theorems[4] on monotone functions. The complex-valued analytic case (Dirichlet Series) has been treated by Jessen[5] under the assumption of linearly independent frequencies, a restriction essential for his method. By the use of Radon's theory of set functions[6] we obtain in the present paper the generalization in the complex domain of Wintner's results by a method analogous to that of his second paper.[3] An elaboration of the present paper, containing extensions and applications of these results, will be given later.

Let $\varphi(E)$ be a monotone absolutely additive set function[7] such that $\varphi(E) = 0$ for all sets E lying outside a fixed rectangle $J:(-M < x < M; -M < y < M)$ and let $F(x, y)$ be the corresponding point function.[8] Radon has shown[9] that F is then continuous in x and y together in the entire plane save at most on a denumerable set of lines $x = \xi_j$ and $y = \eta_i$ which we shall designate as singular lines. In addition to these properties, we shall use

THEOREM I. If $F^{I}(x, y)$, $F^{II}(x, y)$ have the properties of F above and if $F^{I}(x, y)$ is continuous at a point (ξ, η) and if, finally, $F^{I}(x, y) = F^{II}(x, y)$ at a set of points everywhere dense in the plane, then (a) $F^{II}(x, y)$ is continuous at (ξ, η), and (b) $F^{I}(\xi, \eta) = F^{II}(\xi, \eta)$.

PROOF: (a) Suppose the contrary. Then there would exist a sequence of points (x_i, y_i) such that $\lim(x_i, y_i) = (\xi, \eta)$ while either $\lim F^{II}(x_i, y_i)$ did not exist or, if it existed, did not equal $F^{I}(\xi, \eta)$. In either case, $F^{II}(x, y)$ would possess a discontinuity of the second kind, a situation precluded by the monotone character of the function. (b) As F^{II} is continuous at (ξ, η) and $F^{II} = F^{I}$ by hypothesis on a set of points (x_i, y_i) converging to (ξ, η), it follows as in (a) that $F^{II}(\xi, \eta) = F^{I}(\xi, \eta)$.

It follows from Theorem I that the singular lines of $F^{II}(x, y)$ must be the same as those of $F^{I}(x, y)$. We define the singular lines of a monotone absolutely additive set function $\varphi(E)$ to be the singular lines of the corresponding point function $F(x, y)$. A rectangle with sides parallel to the coördinate axes is said to be non-singular if none of its sides lies on a singular line. Furthermore, we shall say that a sequence of rectangles $\{R_n\}$ is

everywhere dense if for every given rectangle $R:(x^1 < x < x^2;\ y^1 < y < y^2)$, not necessarily belonging to the sequence, and for every $\epsilon > 0$ there exists an n such that for the corresponding R_n we have $|x^1-x_n^1|$, $|x^2-x_n^2|$, $|y^1-y_n^1|$, $|y^2-y_n^2|$ all less than ϵ.

From these definitions and from Theorem I we obtain immediately

THEOREM II. If φ^{I} and φ^{II} be two monotone absolutely additive set functions such that $\varphi^{\mathrm{I}}(E) = \varphi^{\mathrm{II}}(E) = 0$ for all E outside a fixed rectangle J and that $\varphi^{\mathrm{I}}(R_n) = \varphi^{\mathrm{II}}(R_n)$ where the sequence $\{R_n\}$ is everywhere dense, then $\varphi^{\mathrm{I}}(E) = \varphi^{\mathrm{II}}(E)$ for every rectangle non-singular with respect to φ^{I} and φ^{II}. Furthermore, φ^{I} and φ^{II} have the same singular lines, hence the same singular rectangles.

Radon has shown[7] that an arbitrary absolutely additive set function φ can be expressed as the difference of two monotone absolutely additive set functions φ_1 and φ_2 corresponding to the total positive variation and the total negative variation of the function, respectively. Thus $\varphi(E) = \varphi_1(E)-\varphi_2(E)$. To the monotone set functions φ_1, φ_2 there correspond the point functions F_1, F_2. Then $F(x, y)$, the point function corresponding to $\varphi(E)$, may be seen immediately to be equal to $F_1(x, y)-F_2(x, y)$. It follows that the non-singular lines of F_1 and F_2 are non-singular lines of F and hence by definition non-singular lines of φ, and from the definition[7] of φ_1, φ_2 it follows conversely that the only non-singular lines of F_1 and F_2 are the non-singular lines of F. We are now in a position to prove the important

UNIQUENESS THEOREM (III).[10] Let there be given an absolutely additive set function $\varphi(E)$ such that $\varphi(E) = 0$ if E lies outside a fixed rectangle J and $\int\int x^n y^m d_{xy}\varphi(E) = 0$ for all non-negative integral values of n, m.[11] [Here and hereafter the region $(-\infty < x < \infty;\ -\infty < y < \infty)$ is meant when no integration domain is indicated; there is here no question of the convergence of an infinite integral, since from the restrictions which we shall always impose on our set functions the integral is over only a finite domain J.] Then $\int\int_R d_{xy}\varphi(E) = 0$ for every non-singular rectangle R.

PROOF: In the first place, if $e:(x_1 \leqq x < x_2;\ y_1 \leqq y < y_2)$ be a rectangle whose side $y = y_1$ is a non-singular line of φ, then[12] $\lim_{y = y_1} \int\int_e |d_{xy}\varphi(E)| = 0$. For if $\varphi(E)$ is monotone, this limit becomes

$$\lim_{y = y_1} \int\int d_{xy}\varphi(E) = \lim_{y = y_1} [F(y, x_2)-F(y_1, x_2)-F(y, x_1) + F(y_1, x_1)] \leqq 2 \lim_{y = y_1} [F(y, M)-F(y_1, M)] = 0$$

where $x = M$, $M \geqq x_2 > x_1$, is a non-singular line of φ. On the other hand, if $\varphi(E)$ is not monotone, it may be decomposed into its canonical

monotone components[7] so that $\varphi(E) = \varphi_1(E) - \varphi_2(E)$. If $y = y_1$ be a non-singular line for both φ_1 and φ_2, it follows from the easily demonstrated relation

$$0 \leqq \int\int_e |d_{xy}\varphi(E)| \leqq \int\int_e d_{xy}\varphi_1(E) + \int\int_e d_{xy}\varphi_2(E)$$

that $\lim_{y=y_1} \int\int_e |d_{xy}\varphi(E)| = 0$. A precisely similar result holds if rôles of x and y be interchanged.

From the hypothesis that $\int\int x^n y^m d_{yx}\varphi(E) = 0$ for all non-negative integral values of n, m, it follows by the Weierstrass Approximation Theorem[13] that $\int\int f(x, y) d_{xy}\varphi(E) = 0$ for all functions continuous in x, y together in a fixed rectangle containing J. In particular, we consider the continuous functions $f_n(x, y)$ defined as follows: Let $R: (x_1 \leqq x < x_2;\ y_1 \leqq y < y_2)$ be the non-singular rectangle referred to in the statement of the theorem. Let S_n be the rectangle $(x_1 - n^{-1} \leqq x < x_2 + n^{-1};\ y_1 - n^{-1} \leqq y < y_2 + n^{-1})$ where n is a positive integer. In R and on its boundary, $f_n(x, y) \equiv 1$. Without S_n let $f_n(x, y) \equiv 0$. In $S_n - R$, $f_n(x, y)$ shall be represented by that point of the truncated pyramid having S_n as base and R as top whose projection is (x, y). Then

$$\int\int f_n(x, y) d_{xy}\varphi(E) = \int\int_R d_{xy}\varphi(E) + \sum_{k=1}^{4} \int\int_{e_{nk}} f_n(x, y) d_{xy}\varphi(E)$$

where

$e_{n1}: (x_2 \leqq x < x_2 + n^{-1};\ y_1 \leqq y < y_2 + n^{-1})$
$e_{n2}: (x_1 - n^{-1} \leqq x < x_2;\ y_2 \leqq y < y_2 + n^{-1})$
$e_{n3}: (x_1 - n^{-1} \leqq x < x_1;\ y_1 - n^{-1} \leqq y < y_2)$
$e_{n4}: (x_1 \leqq x < x_2 + n^{-1};\ y_1 - n^{-1} \leqq y < y_1)$.

The integral on the left is zero by hypothesis.

$$\left| \int\int_{e_{nk}} f_n(x, y) d_{xy}\varphi(E) \right| \leqq \int\int_{e_{nk}} |d_{xy}\varphi(E)| < \epsilon/4,\ n > N,\ k = 1, 2, 3, 4,$$

as shown in the first part of the theorem. Hence $|\int\int_R d_{xy}\varphi(E)| < \epsilon$, for $n > N$, so $\int\int_R d_{xy}\varphi(E) = 0$, q. e. d.

There are two theorems given by Radon[14] which are of sufficient importance for our present work to warrant our stating them here, especially as Radon, by combining them, gives them in a somewhat different form. They are as follows:

THE COMPACTNESS THEOREM. From a continuum of monotone absolutely additive set functions $\varphi_T(E)$ of uniformly bounded variation it is possible to select a sequence $\{\varphi_n(E)\}$ such that as n becomes infinite $\varphi_n(R)$ approaches the limit $\varphi(R)$, where $\varphi(E)$ is a set function with the same properties as $\varphi_n(E)$, and R any non-singular rectangle of $\varphi(E)$.

THE TERM BY TERM INTEGRATION THEOREM. Let $\varphi_1, \varphi_2, \ldots$ be a sequence of monotone absolutely additive set functions of uniformly bounded variation in $\bar{J}:(-M \leqq x \leqq M;\ -M \leqq y \leqq M)$ converging to the monotone absolutely additive set function φ on every non-singular rectangle of the latter. Then if E_0 be any closed set within J and $f(x, y)$ be continuous on E_0,

$$\lim_{n=\infty} \int\int_{E_0} f(x, y) d_{xy}\varphi_n(E) = \int\int_{E_0} f(x, y) d_{xy}\varphi(E).$$

We are now in a position to prove our

MOMENTUM THEOREM (IV). Let there be given a continuum of monotone absolutely additive set functions φ_T of uniformly bounded variation and such that $\varphi_T = 0$ for all T and for all E outside a fixed rectangle J. Then if as T becomes infinite $\lim \int\int x^n y^m d_{xy}\varphi_T(E) = \mu_{nm}$ for all non-negative integers m, n, $\lim \varphi_T(R) = \varphi(R)$, where $\varphi(E)$ has the same properties as $\varphi_T(E)$ and R is any rectangle non-singular with respect to $\varphi(E)$.

PROOF: It follows from the Compactness Theorem that there exists a sequence of functions $\{\varphi_r^{\mathrm{I}}\}$ such that $\lim \varphi_r^{\mathrm{I}}(R) = \varphi^{\mathrm{I}}(R)$, where φ^{I} possesses the same properties as φ_T. If it were possible to select a second sequence $\{\varphi_r^{\mathrm{II}}\}$ converging to $\varphi^{\mathrm{II}}(R)$, we could apply the Term by Term Integration Theorem to

$$\lim_{r=\infty} \int\int x^n y^m d_{xy}\varphi_r^{\mathrm{I}}(E) = \lim_{r=\infty} \int\int x^n y^m d_{xy}\varphi_r^{\mathrm{II}}(E)$$

obtaining

$$\int\int x^n y^m d_{xy}\varphi^{\mathrm{I}}(E) = \int\int x^n y^m d_{xy}\varphi^{\mathrm{II}}(E)$$

or $\int\int x^n y^m d_{xy}\psi(E) = 0$, where $\psi = \varphi^{\mathrm{I}} - \varphi^{\mathrm{II}}$ is an absolutely additive set function. It follows from our Uniqueness Theorem that $\varphi^{\mathrm{I}}(R) = \varphi^{\mathrm{II}}(R)$ on all rectangles R non-singular with respect to ψ, φ^{I} and φ^{II}, and hence on an everywhere dense set of rectangles in the plane. Then by Theorem II it follows that $\varphi^{\mathrm{I}}(R) = \varphi^{\mathrm{II}}(R)$ on every rectangle non-singular with respect to $\varphi^{\mathrm{I}} = \varphi^{\mathrm{II}}$. On placing $\varphi = \varphi^{\mathrm{I}}$, we obtain $\lim \varphi_T(R) = \varphi(R)$ on all non-singular rectangles of φ, q. e. d.

We next prove

THEOREM V. Let $z(t) = x(t) + iy(t)$ be a single-valued complex function of the real variable t defined in $-\infty < t < \infty$, bounded there and measurable, in the sense that $x(t)$, $y(t)$ are both measureable, in every finite interval. If E be any measurable set in the z-plane, T a positive number and (E, T) the set of those points t for which $-T \leqq t \leqq T$ and $z(t)$ belongs to E, we define the monotone absolutely additive set function $\varphi_T(E) = (1/2T)$ meas (E, T). Then

$$\frac{1}{2T}\int_{-T}^{T}[x(t)]^n[y(t)]^m dt = \int\int x^n y^m d_{xy}\varphi_T(E). \qquad (1)$$

PROOF: As t varies over $[-T, T]$, the point (x, y) will vary over a certain measureable set E^* which is bounded by hypothesis. The Lebesgue integral on the left of (1) may be written as[15]

$$n \; sup \sum_{\nu=1}^{p} \eta_{\nu-1}\,\text{meas}\{\eta_{\nu-1} \leqq f(t) < \eta_\nu\} =$$

$$fin\; inf \sum_{\nu=1}^{p} \eta_\nu\,\text{meas}\{\eta_{\nu-1} \leqq f(t) < \eta_\nu\}$$

where $A = \eta_0 < \eta_1 \ldots < \eta_p = B$ is a subdivision of $[A, B]$, within which $f(t)$ lies, for all possible subdivisions such that $|\eta_\nu - \eta_{\nu-1}| \longrightarrow 0$, $\nu = 1, 2, \ldots p$, as $p \longrightarrow \infty$. Under these conditions,

$$\lim \sum_{\nu=1}^{p} x_\nu^n y_\nu^m\,\text{meas}\{\eta_{\nu-1} \leqq f(t) < \eta_\nu\} \qquad (2)$$

exists and is equal to the value of the integral, $x_\nu^n y_\nu^m$ being a value of $f(t)$ in $[\eta_{\nu-1}, \eta_\nu]$ if such exists; otherwise $\text{meas}\{\eta_{\nu-1} \leqq f(t) < \eta_\nu\} = 0$ so the definition is not affected. Again

$$\text{meas}\{\eta_{\nu-1} \leqq f(t) < \eta_\nu\} = \sum_{i=1}^{r}\sum_{j=1}^{s}\text{meas}\begin{Bmatrix}\eta_{\nu-1} \leqq f(t) < \eta_\nu \\ x_{i-1} \leqq x < x_i \\ y_{j-1} \leqq y < y_j\end{Bmatrix} =$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{s}\text{meas}\{R_{\nu ij}\}$$

since the sets on the right are d sjoint, the $[x_{i-1}, x_i)$[16] representing a division of $[A_1, B_1)$ into r equal parts and the $[y_{j-1}, y_j)$ a similar division of $[A_2, B_2)$, where all points z lie in $F: (A_1 \leqq x < B_1;\ A_2 \leqq y < B_2)$. Then (2) may be written

$$\lim \sum_{\nu=1}^{p}\sum_{i=1}^{r}\sum_{j=1}^{s} x_{\nu ij}^n y_{\nu ij}^m\,\text{meas}\{R_{\nu ij}\},\ \text{where}\ (x_{\nu ij}, y_{\nu ij}) \subset R_{\nu ij}. \qquad (3)$$

Now $\sum_{\nu=1}^{p}\sum_{i=1}^{r}\sum_{j=1}^{s} R_{\nu ij} = F$ and $\lim_{p,r,s,=\infty} \delta(R_{\nu ij}) = 0$. Furthermore, $x^n y^m$ is uniformly continuous in $(A_1 \leqq x \leqq B_1;\ A_2 \leqq y \leqq B_2)$ and $(1/2T)$ meas (E, T) is a monotone absolutely additive set function defined there. Therefore the Radon integral on the right hand of (1) exists. Since it is independent of the mode of subdivision of F provided only the latter be sufficiently fine,[11] it follows that this integral is given by (3) divided by $2T$, thus proving (1).

Our final result is given by

THEOREM VI. (*a*) If $z(t) = x(t) + iy(t)$ be an almost periodic function

and $\varphi_T(E)$ the set function defined in Theorem V, there exists a monotone absolutely additive set function $\varphi(E)$ such that as T becomes infinite $\lim \varphi_T(E) = \varphi(E)$ for all non-singular rectangles of φ and $\varphi(E) = 0$ for all sets E lying outside a fixed rectangle J. (*b*) Furthermore, $\mathfrak{M}(x^n y^m) = \mathrm{M}_{nm}(\varphi)$ where $\mathfrak{M}(\ldots) = \lim_{T=\infty} \frac{1}{2T} \int_{-T}^{T} \ldots dt$ and $\mathrm{M}_{nm}(\varphi) = \int\int x^n y^m d_{xy}\varphi(E)$, the mnth momentum of φ. $\varphi(E)$ is uniquely determined up to an additive constant by the momenta save on its singular lines.

PROOF: Since $z(t) = x(t) + iy(t)$, and therefore $x(t)^n y(t)^m$, is almost-periodic, the time momenta $\mathfrak{M}(x^n y^m)$ exist and all functions $x(t)^n y(t)^m$ are bounded and continuous.[17] Therefore, by Theorem V, as T becomes infinite $\lim \int\int x^n y^m d_{xy}\varphi_T(E)$ exists. It follows from the Momentum Theorem that $\lim \varphi_T(R) = \varphi(R)$ for all rectangles R non-singular with respect to φ. Since the introduction of a continuous limit introduces no change in the Term by Term Integration Theorem,

$$\lim_{T=\infty} \int\int x^n y^m d_{xy}\varphi_T(E) = \int\int x^n y^m d_{xy}\varphi(E)$$

so $\mathfrak{M}(x^n, y^m) = \mathrm{M}_{nm}(\varphi)$, where it follows from the Uniqueness Theorem that φ is completely determined save on its singular lines up to an additive constant, q. e. d.

As $\varphi(E)$ is precisely the repartition function belonging to $z(t)$,[18] we have therefore shown that every almost-periodic function possesses a repartition function.

[1] Cf. H. Bohr, *Fastperiodische Funktionen*, Berlin, 28 (1932).

[2] A. Wintner, *Math. Zeitschr.*, **30**, 290–319 (1929).

[3] A. Wintner, *Ibid.*, **36**, 618–629 (1933).

[4] E. Helly, *Wiener Sitzungsber.*, **121**, 265–297 (1912).

[5] B. Jessen, *Bidrag til Integralteorien for Funktioner af uendelig mange Variable*, Copenhagen (1930). The more restricted assumption of Jessen makes it possible for him to infer the existence and continuity of the derivative F_{xy}.

[6] J. Radon, *Wiener Sitzungsber.*, **122**, 1295–1438 (1913), more particularly 1298–1322, and **128**, 1083–1121 (1919).

[7] J. Radon, loc. cit., 1298–1303. For a general treatment of set functions, cf. H. Hahn, *Reelle Funktionen*, Berlin, 393–424 (1921).

[8] J. Radon, loc. cit., p. 1304.

[9] J. Radon, loc. cit., p. 1093.

[10] In the one-dimensional case, this corresponds to the well-known theorem of Lerch.

[11] For double Stieltjes integrals cf. J. Radon, loc. cit., pp. 1322–1324.

[12] For the definition of this symbol, see Radon, loc. cit., p. 1302.

[13] Cf. E. W. Hobson, *Functions of a Real Variable*, 2nd Ed., vol. II, pp. 230–232. The Weierstrass Theorem holds, of course, only for finite regions in the xy-plane, but, as has been pointed out, our integral is extended effectively over a finite domain.

[14] The proof for these theorems is to be found in the article of Radon, loc. cit., pp. 1337–1342, with simplifications in loc. cit., pp. 1092–1094. In the former article, p. 1338, a step is missing, viz., the proof that $F^*(x-h, y-k) \longrightarrow F^*(x, y)$ when $h, k \longrightarrow$

$+ 0$ and (x, y), $(x-h, y-k)$ belong to the everywhere dense set of points. The elaborated proof of this point will be given in my paper mentioned above.

[15] For the notation, see E. Kamke, *Das Lebesguesche Integral*, Leipzig and Berlin, 97 (1925).

[16] For the notation, see H. Hahn, op. cit., p. 29.

[17] Cf. H. Bohr, op. cit., Theorem II, p. 30, Theorem IV, p. 33 and Mean Value Theorem, p. 34.

[18] Cf. the definition given in the one-dimensional case by Wintner, loc. cit., p. 619.

SYLOW SUBGROUPS AND THE NUMBER OF OPERATORS WHOSE ORDERS ARE POWERS OF THE SAME PRIME

BY G. A. MILLER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS

Communicated March 20, 1933

If the order of a group G is divisible by p^m but not by p^{m+1}, p being a prime number, the number of its subgroups of order p^m is always of the form $1 + kp$ and these subgroups are called Sylow subgroups of G. Every operator of G whose order is a power of p is known to be contained in at least one of these $1 + kp$ subgroups and hence a necessary and sufficient condition that G contains exactly p_m operators whose orders are powers of p is that $k = 0$. Little is known in regard to the number of these operators when $k > 0$. The object of the present article is to prove a few additional theorems relating to this number.

Suppose that H is a given Sylow subgroup of order p^m contained in G. Every operator of G whose order is a power of p is conjugate under G with some operator of H and the number of these different operators is the sum obtained by multiplying every operator of H by the quotient obtained by dividing $1 + kp$ by the number of the conjugates of H in which this operator appears and then adding these fractions. For instance, if we let H be the group of the square contained in the group of the cube then $p = 2$ and $1 + kp = 3$. Four of the eight given fractions are then equal to unity while each of the other four is equal to 3. The total number of the operators whose orders are powers of 2 which appear in this group is therefore equal to 16 as is well known from other considerations.

Let K_1 represent the cross-cut of H and some other Sylow subgroup H_1 of order p^m contained in G. Since every proper subgroup of a prime power group is invariant under a larger subgroup thereof it results that H involves operators which transform K_1 into itself but do not transform H_1 into itself. The powers of such an operator will transform H_1 into $k_1 p$ subgroups which separately involve K_1 and no other operator of H.

If G involves a Sylow subgroup H_2 which has a different cross-cut K_2 with H there will therefore be k_2p such subgroups, etc. Hence all the Sylow subgroups of order p^m contained in G except H appear in distinct sets of k_1p, k_2p, . . ., $k_\lambda p$ where k_1, k_2, . . ., k_λ represent natural numbers, such that each set is composed of all these subgroups which have the same cross-cut with H.

If one of these cross-cuts K_α is non-invariant under H then there are also $k_\alpha p$ Sylow subgroups of order p^m in G which have another of these conjugates for a cross-cut with H. That is, the number of the Sylow subgroups of order p^m contained in G which have for their separate cross-cuts with a given Sylow subgroup of this order a set of conjugative non-invariant subgroups of this Sylow subgroup is always a multiple of p^2. In particular, *if every cross-cut of a fixed Sylow subgroup of order p^m contained in a group G with the other Sylow subgroups of this order is non-invariant under this fixed Sylow subgroup then the number of these Sylow subgroups in G is of the form $1 + kp^2$.* This theorem is illustrated by the Sylow subgroups of order 8 in the transitive group of degree 6 and order 72 since each one of these subgroups has a non-invariant cross-cut of order 2 with the other eight such subgroups.

If an invariant subgroup of H appears also in H_1 then this subgroup is transformed into itself by operators of H_1 which do not appear in this invariant subgroup. Hence one of the following two conditions must be satisfied. Either this invariant subgroup is transformed into itself by operators of G whose order is not divisible by p or all the operators of H_1 which transform this subgroup into itself must also appear in H. In particular, if an operator of an abelian Sylow subgroup of a group G is not transformed into itself under G by an operator whose order is prime to the order of this subgroup then it cannot appear in the cross-cut of this Sylow subgroup and another such subgroup. Hence it results that *if a group involves a set of abelian Sylow subgroups such that none of the proper subgroups of one of these Sylow subgroups is transformed into itself by an operator of the group whose order is prime to the order of this subgroup then the cross-cut of every pair of these subgroups is the identity.* This theorem is illustrated by the subgroups of order 4 in the simple group of order 60 since each of the operators of order 2 contained in this group is invariant under only four operators thereof.

The theorems noted above may be employed to determine new properties of the widely used simple group of order 168 as follows: It is known that the octic group is a Sylow subgroup of this simple group and that it has 21 conjugates thereunder. Each of the five operators of order 2 contained in such an octic group H must appear in 5 of these 21 Sylow subgroups since the 21 operators of order 2 in the simple group of order 168 are known to be conjugate thereunder. The invariant operator of order

2 in H is transformed into itself by four of the operators of each of the other four Sylow subgroups in which it appears. Hence the cross-cut of H with each of these four subgroups is of order 4 and non-cyclic. Each of the two non-cyclic subgroups of order 4 contained in H is therefore its cross-cut with two other Sylow subgroups of G and the non-invariant subgroups of order 2 contained in H are separately invariant subgroups of these four Sylow subgroups.

A non-invariant subgroup of order 2 contained in H must therefore constitute the cross-cut of H with two other of its Sylow subgroups since every subgroup of order 4 contained in H includes its invariant subgroup of order 2. Hence H has a cross-cut of order 2 with each of eight of its conjugates and a cross-cut of order 4 with each of four others of its conjugates while its cross-cut with each of the remaining eight conjugates is the identity. The number of the operators of G whose orders are powers of 2 is therefore $1 + {}^{21}/_{5} + {}^{84}/_{5} + 42$. The first of these numbers corresponds to the identity in H, the second corresponds to its invariant operator of order 2, the third corresponds to its four non-invariant operators of order 2, while the last corresponds to its two operators of order 4. Since H is completely determined by each of its operators of order 4 such an operator cannot appear in the cross-cut of H with one of its conjugates.

Suppose that G contains a set of non-abelian Sylow subgroups of order p^m such that every non-invariant operator of one of these subgroups has exactly p conjugates thereunder and that none of the operators besides the identity of this subgroup is transformed into itself under G by an operator which is prime to its order. If such a subgroup H has a non-identity cross-cut involving s with one of its conjugates then s is invariant under a conjugate of H which has a subgroup of index p in common with H. If s is invariant under H this subgroup must be contained in H. This proves the following theorem: *If a group G involves more than one non-abelian Sylow subgroup of order p^m such that none of the non-identity operators involved in one of these subgroups is transformed into itself under G by an operator which is prime to p and that the non-invariant operators of one of these subgroups have only p conjugates thereunder then each operator of the cross-cut of two of these Sylow subgroups is invariant under one of their conjugates and two of the subgroups have a cross-cut of index p under each of them which is transformed into itself by operators of G which are prime to p unless the cross-cut of every pair of these subgroups is the identity.*

Since an operator whose order is a power of p cannot transform a Sylow subgroup of order p^m into itself unless it is contained therein it results that H must transform its conjugates, besides itself, according to a permutation group whose transitive constituents have for their degrees powers of p which exceed the zero power. A necessary and sufficient condition that such a transitive constituent is regular is that the cross-

cut of H with the corresponding Sylow subgroups is invariant under H. This results directly from the theorem that *the degree of the group according to which H transforms transitively a set of its conjugates under G is equal to the index under H of its cross-cuts with one of these conjugates and the order of this transitive group* is equal to the product of its degree and *the index under this cross-cut of the largest invariant subgroup of H contained therein.*

The preceding theorems imply a linear equation whose three unknowns are the number of the Sylow subgroups of order p^m contained in a group G, the number of such subgroups in which a given operator appears and the number of these operators which are contained in G. If two of these numbers are given the third can be found by solving a linear equation in one unknown. For instance, if G represents the symmetric group of degree $2p$, $p > 2$, it obviously contains intransitive Sylow subgroups of order p^2 involving separately $2p - 2$ cyclic permutations of degree p. Such a given cyclic permutation appears in $(p - 2)!$ of these Sylow subgroups since this is the number of these subgroups in the symmetric group of degree p and hence this number is of the form $1 + kp$. The number of such cyclic permutations in G is clearly equal to

$$2(2p - 1)(2p - 2) \ldots (p + 1).$$

It was noted above that this number is the sum of the products obtained by multiplying each of the $2p - 2$ cyclic permutations of order p in one of the Sylow subgroups by the number of the Sylow subgroups of G and dividing this sum by $(p - 2)!$ Hence it results that the number of Sylow subgroups of order p^2 contained in G is

$$2(2p - 1)(2p - 2) \ldots (p + 1)(p - 2)!/(2p + 2).$$

Each of the permutations of order p and of degree $2p$ appears in only one of the Sylow subgroups of order p^2 and such a subgroup contains $(p - 1)^2$ permutations of degree $2p$. Hence the total number of permutations of order p in this symmetric group is

$$(2p - 1)(2p - 2) \ldots (p + 1)[2 + (p - 1)(p - 2)!].$$

These results may also be found directly and are given here simply to illustrate the theorems developed above.

VALUES OF T^{ν}_{μ} AND THE CHRISTOFFEL SYMBOLS FOR A LINE ELEMENT OF CONSIDERABLE GENERALITY

BY HERBERT DINGLE

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, AND IMPERIAL COLLEGE OF SCIENCE AND TECHNOLOGY, ENGLAND

Communicated March 20, 1933

In the general theory of relativity the mechanical properties of any region of the universe are expressed by the energy-momentum tensor, T^{ν}_{μ}, which is itself calculable from the form of the line element, $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$, applicable to that region. The expressions for T^{ν}_{μ} in the most general case, in which all the $g_{\mu\nu}$ are arbitrary functions of the four coördinates, x^1, x^2, x^3, x^4, are exceedingly complicated, but considerable simplification is introduced if it is assumed that $g_{\mu\nu}(\mu \neq \nu) = 0$. The resulting line element still possesses a large amount of generality, and in the applications of the theory particular forms of it have, in fact, usually been employed. It therefore seems desirable to publish the general expressions for the energy-momentum tensor corresponding to this line element, and it is the purpose of this paper to give them, together with the associated values of the Christoffel symbols of the second kind, in the form best suited for application. The calculations, which are somewhat long, have kindly been checked by Mr. C. C. Steffens of the California Institute of Technology, and the proofs have been carefully read, so that the results may be used with considerable confidence. It is hoped that their publication will save labor for those working in this field.

The expression for the line element is taken as

$$ds^2 = -A(dx^1)^2 - B(dx^2)^2 - C(dx^3)^2 + D(dx^4)^2,$$

where A, B, C and D are any functions of x^1, x^2, x^3 and x^4. Mathematically these functions may be positive or negative, real or imaginary,* but in ordinary applications, in which x^4 is the time-like coördinate, they will clearly always be positive and real. The non-vanishing components of the metric tensor and its contravariant associate are obviously as follows:

$$g_{11} = -A;\ g_{22} = -B;\ g_{33} = -C;\ g_{44} = +D$$

$$g^{11} = -\frac{1}{A};\ g^{22} = -\frac{1}{B};\ g^{33} = -\frac{1}{C};\ g^{44} = +\frac{1}{D};$$

and the determinant, g, is $-ABCD$.

Christoffel Symbols.—These are defined by the expression

$$\{\mu\nu, \sigma\} = \tfrac{1}{2} g^{\sigma\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} + \frac{\partial g_{\nu\lambda}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}\right).$$

* It is assumed that they possess first and second differential coefficients with respect to each of the coördinates.

Their values are

$$\begin{array}{llll}
\{11,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^1} & \{21,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^2} & \{31,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^3} & \{41,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^4} \\
\{11,2\} = -\frac{1}{2B}\frac{\partial A}{\partial x^2} & \{21,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^1} & \{31,2\} = 0 & \{41,2\} = 0 \\
\{11,3\} = -\frac{1}{2C}\frac{\partial A}{\partial x^3} & \{21,3\} = 0 & \{31,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^1} & \{41,3\} = 0 \\
\{11,4\} = +\frac{1}{2D}\frac{\partial A}{\partial x^4} & \{21,4\} = 0 & \{31,4\} = 0 & \{41,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^1} \\
\{12,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^2} & \{22,1\} = -\frac{1}{2A}\frac{\partial B}{\partial x^1} & \{32,1\} = 0 & \{42,1\} = 0 \\
\{12,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^1} & \{22,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^2} & \{32,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^3} & \{42,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^4} \\
\{12,3\} = 0 & \{22,3\} = -\frac{1}{2C}\frac{\partial B}{\partial x^3} & \{32,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^2} & \{42,3\} = 0 \\
\{12,4\} = 0 & \{22,4\} = +\frac{1}{2D}\frac{\partial B}{\partial x^4} & \{32,4\} = 0 & \{42,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^2} \\
\{13,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^3} & \{23,1\} = 0 & \{33,1\} = -\frac{1}{2A}\frac{\partial C}{\partial x^1} & \{43,1\} = 0 \\
\{13,2\} = 0 & \{23,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^3} & \{33,2\} = -\frac{1}{2B}\frac{\partial C}{\partial x^2} & \{43,2\} = 0 \\
\{13,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^1} & \{23,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^2} & \{33,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^3} & \{43,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^4} \\
\{13,4\} = 0 & \{23,4\} = 0 & \{33,4\} = +\frac{1}{2D}\frac{\partial C}{\partial x^4} & \{43,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^3} \\
\{14,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^4} & \{24,1\} = 0 & \{34,1\} = 0 & \{44,1\} = +\frac{1}{2A}\frac{\partial D}{\partial x^1} \\
\{14,2\} = 0 & \{24,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^4} & \{34,2\} = 0 & \{44,2\} = +\frac{1}{2B}\frac{\partial D}{\partial x^2} \\
\{14,3\} = 0 & \{24,3\} = 0 & \{34,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^4} & \{44,3\} = +\frac{1}{2C}\frac{\partial D}{\partial x^3} \\
\{14,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^1} & \{24,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^2} & \{34,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^3} & \{44,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^4}
\end{array}$$

Energy-Momentum Tensor, T^{ν}_{μ}.—This tensor is defined by the expression

$$-8\pi T^{\nu}_{\mu} = G^{\nu}_{\mu} - \tfrac{1}{2}\, g^{\nu}_{\mu} G + g^{\nu}_{\mu}\lambda$$

where G^{ν}_{μ} is the contracted Riemann-Christoffel tensor, G is the invariant,

$g^{\mu\nu}G_{\mu\nu}$, and λ is the cosmological constant. The values of $-8\pi T^{\nu}_{\mu}$ are as follows:

$$-8\pi T^1_1 = \frac{1}{2}\left[\frac{1}{BC}\left(\frac{\partial^2 B}{\partial(x^3)^2}+\frac{\partial^2 C}{\partial(x^2)^2}\right)-\frac{1}{BD}\left(\frac{\partial^2 B}{\partial(x^4)^2}-\frac{\partial^2 D}{\partial(x^2)^2}\right)-\frac{1}{CD}\left(\frac{\partial^2 C}{\partial(x^4)^2}-\frac{\partial^2 D}{\partial(x^3)^2}\right)\right]$$
$$-\frac{1}{4}\left[\frac{1}{BC^2}\left\{\frac{\partial B}{\partial x^3}\frac{\partial C}{\partial x^3}+\left(\frac{\partial C}{\partial x^2}\right)^2\right\}+\frac{1}{CB^2}\left\{\frac{\partial C}{\partial x^2}\frac{\partial B}{\partial x^2}+\left(\frac{\partial B}{\partial x^3}\right)^2\right\}\right.$$
$$-\frac{1}{BD^2}\left\{\frac{\partial B}{\partial x^4}\frac{\partial D}{\partial x^4}-\left(\frac{\partial D}{\partial x^2}\right)^2\right\}+\frac{1}{DB^2}\left\{\frac{\partial D}{\partial x^2}\frac{\partial B}{\partial x^2}-\left(\frac{\partial B}{\partial x^4}\right)^2\right\}$$
$$-\frac{1}{CD^2}\left\{\frac{\partial C}{\partial x^4}\frac{\partial D}{\partial x^4}-\left(\frac{\partial D}{\partial x^3}\right)^2\right\}+\frac{1}{DC^2}\left\{\frac{\partial D}{\partial x^3}\frac{\partial C}{\partial x^3}-\left(\frac{\partial C}{\partial x^4}\right)^2\right\}$$
$$-\frac{1}{BCD}\left\{\frac{\partial C}{\partial x^2}\frac{\partial D}{\partial x^2}+\frac{\partial B}{\partial x^3}\frac{\partial D}{\partial x^3}-\frac{\partial B}{\partial x^4}\frac{\partial C}{\partial x^4}\right\}-\frac{1}{ABC}\frac{\partial B}{\partial x^1}\frac{\partial C}{\partial x^1}$$
$$\left.-\frac{1}{ABD}\frac{\partial B}{\partial x^1}\frac{\partial D}{\partial x^1}-\frac{1}{ACD}\frac{\partial C}{\partial x^1}\frac{\partial D}{\partial x^1}\right]+$$

$$-8\pi T^2_2 = \frac{1}{2}\left[\frac{1}{AC}\left(\frac{\partial^2 A}{\partial(x^3)^2}+\frac{\partial^2 C}{\partial(x^1)^2}\right)-\frac{1}{AD}\left(\frac{\partial^2 A}{\partial(x^4)^2}-\frac{\partial^2 D}{\partial(x^1)^2}\right)-\frac{1}{CD}\left(\frac{\partial^2 C}{\partial(x^4)^2}-\frac{\partial^2 D}{\partial(x^3)^2}\right)\right]$$
$$-\frac{1}{4}\left[\frac{1}{AC^2}\left\{\frac{\partial A}{\partial x^3}\frac{\partial C}{\partial x^3}+\left(\frac{\partial C}{\partial x^1}\right)^2\right\}+\frac{1}{CA^2}\left\{\frac{\partial C}{\partial x^1}\frac{\partial A}{\partial x^1}+\left(\frac{\partial A}{\partial x^3}\right)^2\right\}\right.$$
$$-\frac{1}{AD^2}\left\{\frac{\partial A}{\partial x^4}\frac{\partial D}{\partial x^4}-\left(\frac{\partial D}{\partial x^1}\right)^2\right\}+\frac{1}{DA^2}\left\{\frac{\partial D}{\partial x^1}\frac{\partial A}{\partial x^1}-\left(\frac{\partial A}{\partial x^4}\right)^2\right\}$$
$$-\frac{1}{CD^2}\left\{\frac{\partial C}{\partial x^4}\frac{\partial D}{\partial x^4}-\left(\frac{\partial D}{\partial x^3}\right)^2\right\}+\frac{1}{DC^2}\left\{\frac{\partial D}{\partial x^3}\frac{\partial C}{\partial x^3}-\left(\frac{\partial C}{\partial x^4}\right)^2\right\}$$
$$-\frac{1}{ACD}\left\{\frac{\partial C}{\partial x^1}\frac{\partial D}{\partial x^1}+\frac{\partial A}{\partial x^3}\frac{\partial D}{\partial x^3}-\frac{\partial A}{\partial x^4}\frac{\partial C}{\partial x^4}\right\}-\frac{1}{ABC}\frac{\partial A}{\partial x^2}\frac{\partial C}{\partial x^2}$$
$$\left.-\frac{1}{ABD}\frac{\partial A}{\partial x^2}\frac{\partial D}{\partial x^2}-\frac{1}{BCD}\frac{\partial C}{\partial x^2}\frac{\partial D}{\partial x^2}\right]+\lambda$$

$$-8\pi T^3_3 = \frac{1}{2}\left[\frac{1}{AB}\left(\frac{\partial^2 A}{\partial(x^2)^2}+\frac{\partial^2 B}{\partial(x^1)^2}\right)-\frac{1}{AD}\left(\frac{\partial^2 A}{\partial(x^4)^2}-\frac{\partial^2 D}{\partial(x^1)^2}\right)-\frac{1}{BD}\left(\frac{\partial^2 B}{\partial(x^4)^2}-\frac{\partial^2 D}{\partial(x^2)^2}\right)\right]$$
$$-\frac{1}{4}\left[\frac{1}{AB^2}\left\{\frac{\partial A}{\partial x^2}\frac{\partial B}{\partial x^2}+\left(\frac{\partial B}{\partial x^1}\right)^2\right\}+\frac{1}{BA^2}\left\{\frac{\partial B}{\partial x^1}\frac{\partial A}{\partial x^1}+\left(\frac{\partial A}{\partial x^2}\right)^2\right\}\right.$$

$$-\frac{1}{AD^2}\left\{\frac{\partial A}{\partial x^4}\frac{\partial D}{\partial x^4}-\left(\frac{\partial D}{\partial x^1}\right)^2\right\}+\frac{1}{DA^2}\left\{\frac{\partial D}{\partial x^1}\frac{\partial A}{\partial x^1}-\left(\frac{\partial A}{\partial x^4}\right)^2\right\}$$
$$-\frac{1}{BD^2}\left\{\frac{\partial B}{\partial x^4}\frac{\partial D}{\partial x^4}-\left(\frac{\partial D}{\partial x^2}\right)^2\right\}+\frac{1}{DB^2}\left\{\frac{\partial D}{\partial x^2}\frac{\partial B}{\partial x^2}-\left(\frac{\partial B}{\partial x^4}\right)^2\right\}$$
$$-\frac{1}{ABD}\left\{\frac{\partial B}{\partial x^1}\frac{\partial D}{\partial x^1}+\frac{\partial A}{\partial x^2}\frac{\partial D}{\partial x^2}-\frac{\partial A}{\partial x^4}\frac{\partial B}{\partial x^4}\right\}-\frac{1}{ABC}\frac{\partial A}{\partial x^3}\frac{\partial B}{\partial x^3}$$
$$-\frac{1}{ACD}\frac{\partial A}{\partial x^3}\frac{\partial D}{\partial x^3}-\frac{1}{BCD}\frac{\partial B}{\partial x^3}\frac{\partial D}{\partial x^3}\Big]+\lambda$$

$$-8\pi T_4^4=\frac{1}{2}\Big[\frac{1}{AB}\left(\frac{\partial^2 A}{\partial(x^2)^2}+\frac{\partial^2 B}{\partial(x^1)^2}\right)+\frac{1}{AC}\left(\frac{\partial^2 A}{\partial(x^3)^2}+\frac{\partial^2 C}{\partial(x^1)^2}\right)$$
$$+\frac{1}{BC}\left(\frac{\partial^2 B}{\partial(x^3)^2}+\frac{\partial^2 C}{\partial(x^2)^2}\right)\Big]$$
$$-\frac{1}{4}\Big[\frac{1}{AB^2}\left\{\frac{\partial A}{\partial x^2}\frac{\partial B}{\partial x^2}+\left(\frac{\partial B}{\partial x^1}\right)^2\right\}+\frac{1}{BA^2}\left\{\frac{\partial B}{\partial x^1}\frac{\partial A}{\partial x^1}+\left(\frac{\partial A}{\partial x^2}\right)^2\right\}$$
$$+\frac{1}{AC^2}\left\{\frac{\partial A}{\partial x^3}\frac{\partial C}{\partial x^3}+\left(\frac{\partial C}{\partial x^1}\right)^2\right\}+\frac{1}{CA^2}\left\{\frac{\partial C}{\partial x^1}\frac{\partial A}{\partial x^1}+\left(\frac{\partial A}{\partial x^3}\right)^2\right\}$$
$$+\frac{1}{BC^2}\left\{\frac{\partial B}{\partial x^3}\frac{\partial C}{\partial x^3}+\left(\frac{\partial C}{\partial x^2}\right)^2\right\}+\frac{1}{CB^2}\left\{\frac{\partial C}{\partial x^2}\frac{\partial B}{\partial x^2}+\left(\frac{\partial B}{\partial x^3}\right)^2\right\}$$
$$-\frac{1}{ABC}\left\{\frac{\partial B}{\partial x^1}\frac{\partial C}{\partial x^1}+\frac{\partial A}{\partial x^2}\frac{\partial C}{\partial x^2}+\frac{\partial A}{\partial x^3}\frac{\partial B}{\partial x^3}\right\}+\frac{1}{ABD}\frac{\partial A}{\partial x^4}\frac{\partial B}{\partial x^4}$$
$$+\frac{1}{ACD}\frac{\partial A}{\partial x^4}\frac{\partial C}{\partial x^4}+\frac{1}{BCD}\frac{\partial B}{\partial x^4}\frac{\partial C}{\partial x^4}\Big]+\lambda$$

$$-8\pi AT_2^1=-8\pi BT_1^2=$$
$$-\frac{1}{2}\Big[\frac{1}{C}\frac{\partial^2 C}{\partial x^1\partial x^2}+\frac{1}{D}\frac{\partial^2 D}{\partial x^1\partial x^2}\Big]$$
$$+\frac{1}{4}\Big[\frac{1}{C^2}\frac{\partial C}{\partial x^1}\frac{\partial C}{\partial x^2}+\frac{1}{D^2}\frac{\partial D}{\partial x^1}\frac{\partial D}{\partial x^2}+\frac{1}{AC}\frac{\partial A}{\partial x^2}\frac{\partial C}{\partial x^1}+\frac{1}{AD}\frac{\partial A}{\partial x^2}\frac{\partial D}{\partial x^1}$$
$$+\frac{1}{BC}\frac{\partial B}{\partial x^1}\frac{\partial C}{\partial x^2}+\frac{1}{BD}\frac{\partial B}{\partial x^1}\frac{\partial D}{\partial x^2}\Big]$$

$$-8\pi AT_3^1=-8\pi CT_1^3=$$
$$-\frac{1}{2}\Big[\frac{1}{B}\frac{\partial^2 B}{\partial x^1\partial x^3}+\frac{1}{D}\frac{\partial^2 D}{\partial x^1\partial x^3}\Big]$$
$$+\frac{1}{4}\Big[\frac{1}{B^2}\frac{\partial B}{\partial x^1}\frac{\partial B}{\partial x^3}+\frac{1}{D^2}\frac{\partial D}{\partial x^1}\frac{\partial D}{\partial x^3}+\frac{1}{AB}\frac{\partial A}{\partial x^3}\frac{\partial B}{\partial x^1}+\frac{1}{AD}\frac{\partial A}{\partial x^3}\frac{\partial D}{\partial x^1}$$
$$+\frac{1}{CB}\frac{\partial C}{\partial x^1}\frac{\partial B}{\partial x^3}+\frac{1}{CD}\frac{\partial C}{\partial x^1}\frac{\partial D}{\partial x^3}\Big]$$

$$-8\pi BT_3^2 = -8\pi CT_2^3 =$$
$$-\frac{1}{2}\left[\frac{1}{A}\frac{\partial^2 A}{\partial x^2 \partial x^3} + \frac{1}{D}\frac{\partial^2 D}{\partial x^2 \partial x^3}\right]$$
$$+\frac{1}{4}\left[\frac{1}{A^2}\frac{\partial A}{\partial x^2}\frac{\partial A}{\partial x^3} + \frac{1}{D^2}\frac{\partial D}{\partial x^2}\frac{\partial D}{\partial x^3} + \frac{1}{AB}\frac{\partial A}{\partial x^2}\frac{\partial B}{\partial x^3} + \frac{1}{AC}\frac{\partial A}{\partial x^3}\frac{\partial C}{\partial x^2}\right.$$
$$\left. + \frac{1}{DB}\frac{\partial D}{\partial x^2}\frac{\partial B}{\partial x^3} + \frac{1}{DC}\frac{\partial D}{\partial x^3}\frac{\partial C}{\partial x^2}\right]$$

$$-8\pi AT_4^1 = +8\pi DT_1^4 =$$
$$-\frac{1}{2}\left[\frac{1}{B}\frac{\partial^2 B}{\partial x^1 \partial x^4} + \frac{1}{C}\frac{\partial^2 C}{\partial x^1 \partial x^4}\right]$$
$$+\frac{1}{4}\left[\frac{1}{B^2}\frac{\partial B}{\partial x^1}\frac{\partial B}{\partial x^4} + \frac{1}{C^2}\frac{\partial C}{\partial x^1}\frac{\partial C}{\partial x^4} + \frac{1}{AB}\frac{\partial A}{\partial x^4}\frac{\partial B}{\partial x^1} + \frac{1}{AC}\frac{\partial A}{\partial x^4}\frac{\partial C}{\partial x^1}\right.$$
$$\left. + \frac{1}{DB}\frac{\partial D}{\partial x^1}\frac{\partial B}{\partial x^4} + \frac{1}{DC}\frac{\partial D}{\partial x^1}\frac{\partial C}{\partial x^4}\right]$$

$$-8\pi BT_4^2 = +8\pi DT_2^4 =$$
$$-\frac{1}{2}\left[\frac{1}{A}\frac{\partial^2 A}{\partial x^2 \partial x^4} + \frac{1}{C}\frac{\partial^2 C}{\partial x^2 \partial x^4}\right]$$
$$+\frac{1}{4}\left[\frac{1}{A^2}\frac{\partial A}{\partial x^2}\frac{\partial A}{\partial x^4} + \frac{1}{C^2}\frac{\partial C}{\partial x^2}\frac{\partial C}{\partial x^4} + \frac{1}{AB}\frac{\partial A}{\partial x^2}\frac{\partial B}{\partial x^4} + \frac{1}{AD}\frac{\partial A}{\partial x^4}\frac{\partial D}{\partial x^2}\right.$$
$$\left. + \frac{1}{CB}\frac{\partial C}{\partial x^2}\frac{\partial B}{\partial x^4} + \frac{1}{DC}\frac{\partial D}{\partial x^2}\frac{\partial C}{\partial x^4}\right]$$

$$-8\pi CT_4^3 = +8\pi DT_3^4 =$$
$$-\frac{1}{2}\left[\frac{1}{A}\frac{\partial^2 A}{\partial x^3 \partial x^4} + \frac{1}{B}\frac{\partial^2 B}{\partial x^3 \partial x^4}\right]$$
$$+\frac{1}{4}\left[\frac{1}{A^2}\frac{\partial A}{\partial x^3}\frac{\partial A}{\partial x^4} + \frac{1}{B^2}\frac{\partial B}{\partial x^3}\frac{\partial B}{\partial x^4} + \frac{1}{AC}\frac{\partial A}{\partial x^3}\frac{\partial C}{\partial x^4} + \frac{1}{AD}\frac{\partial A}{\partial x^4}\frac{\partial D}{\partial x^3}\right.$$
$$\left. + \frac{1}{BC}\frac{\partial B}{\partial x^3}\frac{\partial C}{\partial x^4} + \frac{1}{BD}\frac{\partial B}{\partial x^4}\frac{\partial D}{\partial x^3}\right]$$

ON NON-FACTORABLE TRANSFORMATIONS OF DOUBLE SEQUENCES[1]

BY C. RAYMOND ADAMS

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY

Communicated April 10, 1933

In a recent paper,[2] hereafter referred to as I, we considered the transformation of double sequences

$$(A) \qquad y_{mn} = \sum_{k,l=1}^{m,n} a_{mnkl}\, x_{kl},$$

defined by a four-dimensional "triangular" matrix $||a_{mnkl}||$ ($m, n, k, l = 1, 2, 3, \ldots$; $a_{mnkl} = 0$ for $k > m$ or $l > n$ or both) of real or complex numbers. We showed (Theorem 2 of I) that if (A) is regular for bounded sequences, it is in general regular also for a large class of unbounded sequences.

Special interest attaches to the type of transformation (A) for which we have $a_{mnkl} = a'_{mk} \cdot a''_{nl}$ ($m, n, k, l = 1, 2, 3, \ldots$), a'_{mk} and a''_{nl} being, respectively, the general elements of two triangular matrices; this type we call *factorable*, and we write $A = A' \odot A''$. For a factorable transformation to be regular for bounded sequences it is necessary and sufficient[3] that each factor matrix be a *pure C-matrix* in the sense of Hausdorff[4] (i.e., a matrix defining a simple sequence transformation which is convergence-preserving and regular for null sequences) and together they satisfy the condition $l' \cdot l'' = 1$, where

$$l' = \lim_{m \to \infty} \sum_{k=1}^{m} a'_{mk}, \quad l'' = \lim_{n \to \infty} \sum_{l=1}^{n} a''_{nl}.$$

Concerning factorable transformations we gave in I the following theorem, a particular case of Theorem 2 of I cited above.

THEOREM 1. *A transformation $A = A' \odot A''$, with regular factors, is regular for double sequences of which each row has a bounded A''-transform and each column a bounded A'-transform.*

Simultaneously Lösch[5] established the somewhat stronger

THEOREM 2. *A transformation $A = A' \odot A''$, with regular factors, is regular for double sequences whose A-transforms are bounded.*

This result was improved upon slightly by the present writer[3] and more significantly by Agnew[6] in theorems which may be given the following forms, respectively. It should be observed that since a convergent double sequence is ultimately bounded,[7] the class of sequences specified in Theorem 4 cannot be further enlarged.

THEOREM 3. *A transformation $A = A' \odot A''$, regular for bounded se-*

quences, is regular for double sequences whose A-transforms are bounded.

THEOREM 4. *A transformation $A = A' \odot A''$, regular for bounded sequences, is regular for double sequences whose A-transforms are ultimately bounded.*

The question naturally arises as to whether Theorem 2 of I can be improved upon in the same manner in which its particular case, Theorem 1, has been successively strengthened in Theorems 2, 3 and 4. The purpose of this note is to demonstrate, by means of the example contained in the next paragraph, that this is not possible; in other words, to indicate how essential is the hypothesis of factorability in Theorems 2, 3 and 4.

We consider the transformation (A) defined by the following "triangular" matrix:

$$\left.\begin{array}{ll} a_{mn1l} = \delta_{mn}\delta_{nl}/n; \ \delta_{ii} = 1, \ \delta_{ij} = 0 \text{ for } i \neq j, & \\ a_{mnkl} = 1/(mn) \qquad\qquad (k = 2, 3, 4, \ldots), & \end{array}\right\} \quad (m, n, l = 1, 2, 3, \ldots). \tag{1}$$

It may be verified at once that this non-factorable transformation satisfies the Robison-Kojima[8] set of necessary and sufficient conditions that a transformation (A) be regular for bounded sequences:

$$(\alpha) \qquad \sum_{k,l=1}^{m,n} | a_{mnkl} | < M \qquad (m, n = 1, 2, 3, \ldots),$$

$$(\beta) \qquad \lim_{m,n \to \infty} \sum_{k,l=1}^{m,n} a_{mnkl} = 1,$$

$$(\gamma) \qquad \lim_{m,n \to \infty} a_{mnkl} = 0 \qquad (k, l = 1, 2, 3, \ldots),$$

$$(\delta) \qquad \lim_{m,n \to \infty} \sum_{k=1}^{m} | a_{mnkl} | = 0 \qquad (l = 1, 2, 3, \ldots),$$

$$(\epsilon) \qquad \lim_{m,n \to \infty} \sum_{l=1}^{n} | a_{mnkl} | = 0 \qquad (k = 1, 2, 3, \ldots).$$

Applying the transformation defined by (1) to the null sequence

$$x_{1l} = l, \quad x_{kl} = 0 \ (k = 2, 3, 4, \ldots), \quad (l = 1, 2, 3, \ldots),$$

we find that its transform is

$$y_{mn} = \delta_{mn} \qquad (m, n = 1, 2, 3, \ldots),$$

a bounded but divergent sequence. Hence *Theorems 3 and 4 are not valid if the hypothesis of factorability is deleted.*[9]

Now a transformation $A = A' \odot A''$, regular for bounded sequences, satisfies not only the conditions $(\alpha)-(\epsilon)$ but also the set of conditions obtained from them by replacing (δ) and (ϵ), respectively, by the vastly more stringent conditions[10]

(δ') *there exists a constant $M' > 0$, and for each l $(l = 1, 2, 3, \ldots)$ a null sequence $\{b_{nl}\}$ $(n = 1, 2, 3, \ldots)$, such that*

$$\sum_{k=1}^{m} | a_{mnkl} | \leqslant b_{nl} \cdot M' \qquad (m, n = 1, 2, 3, \ldots),$$

and

(ϵ') *there exists a constant* $M'' > 0$, *and for each* k ($k = 1, 2, 3, \ldots$) *a null sequence* $\{c_{mk}\}$ ($m = 1, 2, 3, \ldots$), *such that*

$$\sum_{l=1}^{n} | a_{mnkl} | \leqslant c_{mk} \cdot M'' \qquad (m, n = 1, 2, 3, \ldots).$$

These, we remark incidentally, imply not only (γ) but even

$$(\gamma_1') \qquad | a_{mnkl} | \leqslant b_{nl} \cdot M' \qquad (m, n = 1, 2, 3, \ldots),$$

and

$$(\gamma_2') \qquad | a_{mnkl} | \leqslant c_{mk} \cdot M'' \qquad (m, n = 1, 2, 3, \ldots).$$

Thus the further question is raised as to *whether a non-factorable transformation* (A) *fulfilling the conditions* (α), (β), (δ'), (ϵ') [*and consequently* (γ_1') *and* (γ_2')] *must be regular for sequences having bounded, or perhaps only ultimately bounded,* A*-transforms. The example given above also answers this question in the negative.*

Although the possibility of strengthening Theorem 2 of I is still left open, it would appear to be somewhat dubious.

[1] Presented to the American Mathematical Society, April 14, 1933.

[2] Adams, "Transformations of Double Sequences with Application to Cesàro Summability of Double Series," *Bull. Amer. Math. Soc.*, **37**, 741–748 (1931).

[3] See Adams, "On Summability of Double Series," *Trans. Amer. Math. Soc.*, **34**, 215–230 (1932), Theorems 2 and 3.

[4] Hausdorff, "Summationsmethoden und Momentfolgen, I," *Math. Zeit.*, **9**, 74–109 (1921).

[5] Lösch, "Über den Permanenzsatz gewisser Limitierungsverfahren für Doppelfolgen," *Ibid.*, **34**, 281–290 (1931).

[6] Agnew, "On Summability of Double Sequences," *Amer. Jour. Math.*, **54**, 648–656 (1932). The theorem of Agnew, in slightly less general form, has since been given by Lösch, "Über den Permanenzsatz.... Doppelfolgen, II", *Math. Zeit.*, **37**, 77–84 (1933).

[7] A double sequence $\{x_{mn}\}$ is said to be ultimately bounded if $\limsup_{m,n \to \infty} | x_{mn} |$ is finite.

[8] Robison, "Divergent Double Sequences and Series," *Trans. Amer. Math. Soc.*, **28**, 50–73 (1926); Kojima, "On the Theory of Double Sequences," *Tôhoku Math. Jour.*, **21**, 3–14 (1922). Although Kojima's paper contains no explicit statement of the set of conditions necessary and sufficient for regularity for bounded sequences as obtained by Robison, it does give the related set of conditions necessary and sufficient for convergence-preservation for bounded sequences. For this and other reasons which become apparent to the careful reader of Kojima's paper, we feel that his name should be associated with these regularity conditions.

[9] Clearly the result of replacing the hypothesis of factorability with regular factors by the hypothesis of regularity for bounded sequences in Theorem 2 is the same as of deleting the hypothesis of factorability from Theorem 3.

[10] It is worthwhile noting that (δ') and (ϵ') are stronger than the conditions: $\lim_{n\to\infty} \sum_{k=1}^{m} |a_{mnkl}| = 0$ uniformly with respect to m, for each l, which implies $\lim_{n\to\infty} a_{mnkl} = 0$ uniformly with respect to m for each pair of values k, l; and the symmetric condition for the sum $\sum_{l=1}^{n} |a_{mnkl}|$. Still more noteworthy is the fact that the transformation (1) satisfies and conditions obtained from (δ') and (ϵ') by freeing $\{b_{nl}\}$ and $\{c_{mk}\}$ from dependence upon l and k, respectively, a pair of conditions which is *not* satisfied by all factorable transformations regular for bounded sequences.

THE METHOD OF SPECTRAL REDUCTION IN CLASSICAL DYNAMICS

By A. KHINTCHINE

MATHEMATICAL INSTITUTE OF THE MOSCOW UNIVERSITY

Communicated April 4, 1933

1. For the last year B. O. Koopman and J. v. Neumann published several papers, where they succeeded in proving a series of general and interesting theorems of classical dynamics by means of a new method based on spectral reduction of unitary operators in the Hilbert space. The object of this note is to show that the most important of these results may be obtained by an elementary spectral analysis without applying the theory of operators.

Koopman and v. Neumann's investigations are based on Stone's theorem of the possibility of spectral reduction of any unitary operator. In our method instead of this one, we take the following important theorem due to S. Bochner:[1] *In order that the function $f(x)$ of the real variable x be representable in the form*

$$f(x) = \int_{-\infty}^{+\infty} e^{i\alpha x}\, dF(\alpha),$$

where $F(\alpha)$ is a non-decreasing function of bounded variation, it is necessary and sufficient that it be positive-definite; this means that it must be bounded and continuous in $(-\infty, +\infty)$ and that the integral

$$\int_a^b \int_a^b f(y-z)\varphi(y)\bar{\varphi}(z)dydz$$

must be non-negative for any real a and b, if $\varphi(x)$ be any continuous complex function of the real variable x.

Let some part of a many dimensional euclidean space having a finite volume V, be in a state of continuous and steady motion, in which it passes

into itself, and the measure of each measurable set of points is preserved. Let E be a measurable set of measure $m(E) = v > 0$ included in this part; denote by E_t the set into which E passes during the time t, and let

$$\mu(t) = m(EE_t);$$

in like manner let us denote by P_t the point into which the point P passes during the time t; let $\lambda(P) = 1$ if $P \subset E$ and $\lambda(P) = 0$ if P not $\subset E$; further on, integration with respect to P will always denote the many dimensional integration extended on the volume V. For any continuous complex function $\varphi(x)$ and for any real numbers a and b we have

$$0 \leqq \int \left| \int_a^b \lambda(P_t)\varphi(t)dt \right|^2 dP =$$
$$\int \left\{ \int_a^b \int_a^b \lambda(P_u)\lambda(P_v)\varphi(u)\bar{\varphi}(v)dudv \right\} dP =$$
$$\int_a^b \int_a^b \varphi(u)\bar{\varphi}(v) \left\{ \int \lambda(P_u)\lambda(P_v)dP \right\} dudv =$$
$$\int_a^b \int_a^b \mu(u-v)\varphi(u)\bar{\varphi}(v)dudv.$$

This shows that $\mu(t)$ is positive-definite; on the basis of Bochner's theorem we can put ($\mu(t)$ being real)

$$\mu(t) = \int_{-\infty}^{+\infty} \cos txdF(x)$$

where $F(x)$ is some non-decreasing function of bounded variation (it is evident that $F(+\infty) - F(-\infty) = \mu(0) = m(E) = v$).

2. The fundamental results of Koopman and v. Neumann can be obtained from the formula (1) by means of the following elementary *Lemma; Let $\psi(x; a, b, \ldots)$ be a continuous function of the real variable x depending on the real parameters $a, b, \ldots$ and having the following properties:*

(1) $|\psi(x; a, b, \ldots)| < C$ *for all* x; $a, b, \ldots$

(2) *if all the parameters $a, b, \ldots$ increase infinitely, there is*

$$\psi(x; a, b, \ldots) \longrightarrow 0$$

uniformly for $|x - \lambda| > \delta$, where δ is an arbitrarily small positive number;

(3) *for $x \longrightarrow \lambda$ and any constant values of the parameters $a, b, \ldots$*

$$\psi(x; a, b, \ldots) \longrightarrow A.$$

Let $F(x)$ be any function of bounded variation in $(-\infty, +\infty)$. Then for $a, b, \ldots \longrightarrow \infty$

$$\phi(a, b, \ldots) = \int_{-\infty}^{+\infty} \psi(x;\ a, b, \ldots)dF(x) \longrightarrow A[F(\lambda + 0) - F(\lambda - 0)].$$

In fact let $\int_{-\infty}^{+\infty} |\, dF(x)\,| = K$, and let $\epsilon > 0$ be arbitrarily small.

Let $\delta > 0$ be so small that

$$\int_{\lambda-\delta}^{\lambda-0} |\, dF(x)\,| < \frac{\epsilon}{4C}, \quad \int_{\lambda+0}^{\lambda+\delta} |\, dF(x)\,| < \frac{\epsilon}{4C},$$

and let $a, b, \ldots$ be so great that

$$|\psi(x;\ a, b, \ldots)| < \frac{\epsilon}{2K} \text{ for } |x-\lambda| \geqq \delta.$$

Then

$$\left| \int_{|x-\lambda|>\delta} \psi dF(x) \right| < \frac{\epsilon}{2},$$

and for any positive number η, $0 < \eta < \delta$,

$$\left| \int_{\eta<|x-\lambda|<\delta} \psi dF(x) \right| < C \int_{\eta<|x-\lambda|<\delta} |\, dF(x)\,| < \frac{\epsilon}{2},$$

and so

$$\left| \phi(a, b, \ldots) - \int_{\lambda-\eta}^{\lambda+\eta} \psi(x;\ a, b, \ldots)dF(\lambda) \right| < \epsilon;$$

and η being arbitrarily small, then in virtue of the property (3) of ψ

$$|\,\phi(a, b, \ldots) - A[F(\lambda + 0) - F(\lambda - 0)]\,| \leqq \epsilon,$$

which proves the lemma.

3. The "quasi-ergodic theorem" as proved by v. Neumann[2] is based on the fact that uniformly for $U > 0$

$$\lim_{T \to \infty} \int \left\{ \frac{I(P, T)}{T} - \frac{I(P, T+U)}{T+U} \right\}^2 dP = 0,$$

where

$$I(P, T) = \int_0^T \lambda(P_t)dt.$$

Let us put

$$D(T, U) = \int \left\{ \frac{I(P, T)}{T} - \frac{I(P, T+U)}{T+U} \right\}^2 dG;$$

we find immediately

$$D(T, U) = \frac{1}{T^2(T+U)^2}\int\left\{U\int_0^T \lambda(P_t)dt - T\int_T^{T+U}\lambda(P_t)dt\right\}^2 dP$$

$$= \frac{1}{T^2(T+U)^2}\int\left\{U^2\int_0^T\int_0^T \lambda(P_u)\lambda(P_v)dudv + T^2\int_T^{T+U}\int_T^{T+U}\lambda(P_u)\lambda(P_v)dudv - 2UT\int_0^T\int_T^{T+U}\lambda(P_u)\,\lambda\,(P_v)\,dudv\right\}dP,$$

whence, altering the order of integration and remarking that

$$\int\lambda(P_u)\lambda(P_v)dP = \mu(u-v),$$

we find

$$D(T, U) = \frac{1}{T^2(T+U)^2}\left\{U^2\int_0^T\int_0^T\mu(u-v)dudv + T^2\int_0^U\int_0^U\mu(u-v)dudv - 2UT\int_0^T\int_T^{T+U}\mu(u-v)dudv\right\}.$$

Putting here on the basis of (1) $\mu(u-v) = \int_{-\infty}^{+\infty}\cos[(u-v)x]dF(x)$, we find after some elementary transformations

$$D(T, U) = \frac{U^2}{(T+U)^2}\int_{-\infty}^{+\infty}\left\{\left(\frac{\sin\frac{Tx}{2}}{\frac{Tx}{2}}\right)^2 + \left(\frac{\sin\frac{Ux}{2}}{\frac{Ux}{2}}\right)^2 - 2\frac{\sin\frac{Tx}{2}}{\frac{Tx}{2}}\frac{\sin\frac{Ux}{2}}{\frac{Ux}{2}}\cos\frac{(T+U)x}{2}\right\}dF(x).$$

If as $T \longrightarrow \infty$ the quantity U remains bounded, then

$$\frac{U^2}{(T+U)^2} \longrightarrow 0,$$

and therefore $D(T, U) \longrightarrow 0$; if $U \longrightarrow \infty$, then the integrand possesses all the properties of the function ψ in our lemma, if we put $\lambda = 0$, $A = 0$, and therefore $D(T, U) \longrightarrow 0$ also.

If the volume V cannot be split into two parts of positive measure, each of which remains invariant in the motion in question with precision up to a set of measure zero, then

$$\lim_{T\longrightarrow\infty}\int\left\{\frac{I(P, T)}{T} - \frac{v}{V}\right\}^2 dP = 0.$$

In fact, in virtue of the proved mean-convergence of the ratio $\frac{I(P, T)}{T}$, there exists such a function $f(P)$ that

$$\lim_{T \to \infty} \int \left\{ \frac{I(P, T)}{T} - f(P) \right\}^2 dP = 0;$$

and in this case, as it is known, we may find such a sequence of numbers $T_1 < T_2 < \ldots < T_n < \ldots \to \infty$, that almost everywhere in V

$$\lim_{n \to \infty} \frac{I(P, T_n)}{T_n} = f(P).$$

If the function $f(P)$ were not constant almost everywhere, we could find such a number β that

$$f(P) \geqq \beta \text{ on the set } E_1,\ m(E_1) > 0,$$
$$f(P) < \beta \text{ on the set } E_2,\ m(E_2) > 0;$$

but it is easy to see that the set E_1 (and therefore the set E_2 also) remains invariant in the given motion; in fact, for any I

$$\lim_{n \to \infty} \frac{I(P_\tau, T_n)}{T_n} = \lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} \lambda(P_{\tau + t}) dt$$
$$= \lim_{n \to \infty} \frac{1}{T_n} \int_\tau^{\tau + T_n} \lambda(P_t) dt = \lim_{n \to \infty} \left\{ \frac{1}{T_n} \int_0^{T_n} \lambda(P_t) dt + 0\left(\frac{1}{T_n}\right) \right\} = \lim_{n \to \infty} \frac{I(P, T_n)}{T_n},$$

so that if $P \subset E_1$ then $P_\tau \subset E_1$ too. Hence it follows that almost everywhere

$$f(P) = \lim_{n \to \infty} \frac{I(P, T_n)}{T_n} = \kappa = \text{const.};$$

and as

$$\int \frac{I(P, T_n)}{T_n} dP = \frac{1}{T_n} \int_0^{T_n} dt \int \lambda(P_t) dP = \frac{1}{T_n} \int_0^{T_n} v dt = v,$$

then

$$f(P) = \kappa = \frac{v}{V}.$$

This is the proposition that forms the "quasi-ergodic theorem" proved by v. Neumann.

4. Another interesting result obtained by Koopman and v. Neumann

in their recent joint paper[3] consists principally in that *the function* $\mu(t)$ *may be represented as a sum*

$$\mu(t) = \mu_1(t) + \mu_2(t),$$

where $\mu_1(t)$ *is an almost periodic function and* $\mu_2(t)$ *satisfies the condition*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \{\mu_2(t)\}^2 dt = 0$$

(it is evident that the latter condition is equivalent to $\mu_2(t) \to 0$ for $t \to \infty$ if we neglect a set of values of t having the density 0 at infinity).

Let us put in (1)

$$F(x) = F_1(x) + F_2(x),$$

where $F_1(x)$ has a point spectrum and $F_2(x)$ is continuous. Next put

$$\mu_1(t) = \int_{-\infty}^{+\infty} \cos txdF_1(x), \quad \mu_2(t) = \int_{-\infty}^{+\infty} \cos txdF_2(x),$$

so that

$$\mu(t) = \mu_1(t) + \mu_2(t).$$

It is known that $\mu_1(t)$ is an almost periodic function. On the other hand,

$$\frac{1}{T}\int_0^T \{\mu_2(t)\}^2 dt = \frac{1}{T}\int_0^T \left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cos tx \cos tydF_2(x)dF_2(y)\right\}dt =$$

$$= \frac{1}{2T}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dF_2(y)dF_2(x)\left\{\int_0^T [\cos t(x+y) + \cos t(x-y)]dt\right\} =$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} dF_2(y)\int_{-\infty}^{+\infty}\left\{\frac{\sin T(x+y)}{T(x+y)} + \frac{\sin T(x-y)}{T(x-y)}\right\}dF_2(x);$$

but the integral

$$\int_{-\infty}^{+\infty} \frac{\sin T(x-y)}{T(x-y)} dF_2(x)$$

converges toward $F_2(y+0) - F_2(y-0) = 0$ for $T \to \infty$, in virtue of our lemma; and the same is true for the other integral of the second member; and since $F_2(x)$ is a continuous function with bounded variation in $(-\infty, +\infty)$, this tendency is uniform with respect to y. This gives

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \{\mu_2(t)\}^2 dt = 0.$$

In their work Koopman and v. Neumann consider a more general case, when in the volume V two different measurable sets E_1, E_2 are given, and for the corresponding functions $\lambda_1(P)$ and $\lambda_2(P)$ the integral

$$\nu(t) = \int \lambda_1(P)\lambda_2(P_t)dP$$

representing the measure of the set of points, which pass from E_1 to E_2 during the time t, is examined. Our method easily shows that

$$\nu(t) = \nu_1(t) + \nu_2(t),$$

where $\nu_1(t)$ is an almost periodic function and where

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} \{\nu_2(t)\}^2 dt = 0.$$

[1] S. Bochner, *Vorlesungen über Fouriersche Integrale*, Leipzig, 76, Satz 23 (1932).

[2] J. v. Neumann, "Proof of the Quasi-Ergodic Hypothesis," *Proc. Nat. Acad. Sci.* (U. S. A.), **18**, 70–82 (1932).

[3] B. O. Koopman and J. v. Neumann, "Dynamical Systems of Continuous Spectra," *Proc. Nat. Acad. Sci.* (U. S. A.), **18**, 255–263 (1932).

QUESTIONS OF RELATIVE INCLUSION IN THE DOMAIN OF HAUSDORFF MEANS

BY EINAR HILLE AND J. D. TAMARKIN

PRINCETON UNIVERSITY AND BROWN UNIVERSITY

Communicated March 27, 1933

1. The present note is a preliminary communication concerning the question of relative inclusion of Hausdorff means.[1]

A Hausdorff method of summation, $[H, q(u)]$, is determined by a function $q(u)$ subject to the following conditions:

(i) $q(u)$ is of bounded variation in the closed interval $(0, 1)$;

(ii) $q(u)$ is continuous at $u = 0$ and $q(0) = 0$;

(iii) $q(1) = 1$

(iv) $q(u) = \frac{1}{2}[q(u - 0) + q(u + 0)]$, $0 < u < 1$.

Putting

$$\mu_n = \int_0^1 u^n dq(u), \quad n = 0, 1, 2, \ldots, \tag{1}$$

we form the diagonal matrix $\mathfrak{M}$ whose elements along the main diagonal are $\mu_0, \mu_1, \ldots, \mu_n, \ldots$ Let

$$\mathfrak{D} = \| (-1)^n \binom{m}{n} \|, \quad \mathfrak{H} = \mathfrak{D}\mathfrak{M}\mathfrak{D} = \| h_{mn} \|. \tag{2}$$

Let the vectors $X = (x_0, x_1, \ldots, x_n, \ldots)$, $Y = (y_0, y_1, \ldots, y_n, \ldots)$ be related by the equation .

$$Y = \mathfrak{H}X, \tag{3}$$

i.e.,

$$y_m = \sum_{n=0}^{m} h_{mn}\, x_n,\ m = 0, 1, 2, \ldots \tag{4}$$

we call y_m the m^{th} $[H, q(u)]$-transform of the sequence (x_n). The conditions on $q(u)$ are necessary and sufficient in order that the transformation be regular, i.e., limit preserving. If y_m tends to a finite limit η, the sequence (x_n) is said to be limitable $[H, q(u)]$ to the limit η.

In the following $\mathfrak{H}$ and $\mathfrak{M}$ will be referred to as the *Hausdorff matrix* and the *moment matrix*, respectively, of the method $[H, q(u)]$. Further, $q(u)$ will be called the *mass function*, (μ_n) the *moment sequence*, and

$$\mu(z) = \int_0^1 u^z\, dq(u),\ \Re(z) \geqq 0, \tag{5}$$

the *moment function* of the method. It should be noted that $\mu(z)$ is holomorphic when $\Re(z) > 0$; its absolute value is less than the total variation of $q(u)$ when $\Re(z) \geqq 0$, and it is continuous in the latter region.

2. We recall that a method of summation A is said to *include* another method B, symbolically $A \supset B$, if every sequence limitable B is also limitable A to the same limit. All methods $[H, q(u)]$ are consistent since the corresponding matrices commute.

Hausdorff has given a sufficient condition in order that $[H, q_1(u)] \supset [H, q_2(u)]$. This condition would seem to be necessary as well, but this has been proved only when $\mathfrak{H}_2$ has a unique inverse. The criterion can be given three different but equivalent formulations. (i) There exists a moment sequence (μ_n) such that

$$\mu_{n1} = \mu_n \mu_{n2},\ n = 0, 1, 2, \ldots, \tag{6}$$

the notation being obvious. (ii) There exists a moment matrix $\mathfrak{M}$ and a Hausdorff matrix $\mathfrak{H}$ such that

$$\mathfrak{M}_1 = \mathfrak{M}\mathfrak{M}_2,\ \mathfrak{H}_1 = \mathfrak{H}\mathfrak{H}_2. \tag{7}$$

(iii) There exists a moment function $\mu(z)$ such that

$$\mu_1(z) = \mu(z)\, \mu_2(z). \tag{8}$$

Corresponding to this relation between the moment functions there is a relation between the mass functions, due to R. Schmidt.[2] (iv) There exists a mass function $q(u)$ such that

$$q_1(u) = q(u) + \int_u^1 q_2\left(\frac{u}{v}\right) dq(v). \tag{9}$$

This is an integral equation of the Stieltjes-Volterra type, and need not be

satisfied for all values of u, the exceptional set being, however, of measure zero. If we put

$$u = e^{-s},\ q_\nu(u) = 1 - p_\nu(s),$$

(9) becomes

$$p_1(s) = \int_0^s p_2(s - t)\, dp(t). \tag{10}$$

All these conditions are equivalent.

3. A number of questions concerning relative inclusion has been solved by Hausdorff[3] working with conditions (6)–(8). He used condition (8), however, only for real values of z. We have continued these investigations using conditions (7)–(10). Some special results obtained so far will be stated in this paragraph.

Condition (8) implies that $\mu_1(z)/\mu_2(z)$ is a moment function if $H_1 \supset H_2$. This implies, in particular, that the quotient must be bounded in $\Re(z) \geqq 0$. Here is an application of this observation. Taking H_1 to be the method of Hölder of order α, H^α say, and H_2 to be the method of Euler-Knopp of order p, (E, a) say, where $a = 2^{-p}$, we find that

$$\mu_1(z) = (1 + z)^{-\alpha},\ \mu_2(z) = a^z. \tag{11}$$

It follows that neither $\mu_1(z)/\mu_2(z)$ nor $\mu_2(z)/\mu_1(z)$ is bounded $\Re(z) \geqq 0$. Hence neither method includes the other. This is of course known result. Further, condition (8) shows that a necessary and sufficient condition for H_1 to include (E, a) is that $\mu_1(z) = a^z\mu(z)$. This is equivalent to $q_1(u) = 1$ when $a < u \leqq 1$.

In this connection it is worth remarking that the same simple observation enables us to conclude that none of the functions

$$\exp\,[-A(z + a)^\alpha],\ (1 + z)^{-z},\ 1/\Gamma(z + 1)$$

can be a moment function. In the first example $A > 0$, $a \geqq 0$, $\alpha > 1$, the statement being false for $0 < \alpha \leqq 1$. In the second the powers should be given their principal determinations. The third example is due to Hausdorff.

Condition (8) leads also to another new proof for the equivalence of the means of Cesàro (C, α) with those of Hölder H^α for $\Re(\alpha) > -1$.[4] Using some results of Nörlund[5] we can show that the quotients of the corresponding moment functions can be expanded into factorial series absolutely convergent for $\Re(z) > -1$, $\Re(z + \alpha) > -1$. This implies that the quotients are representable by means of integrals of the type of Laplace which are readily proved to be moment functions in the sense of Hausdorff. Hence the methods are equivalent.

The following example illustrates the use of conditions (9) and (10). In the case in which $H_2 = H^\alpha \approx (C, \alpha)$, (10) becomes

$$e^s p_1(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s - t)^{\alpha-1} e^t p(t)\, dt, \tag{12}$$

i.e., Abel's integral equation. This implies that $e^s p_1(s)$ must be the fractional integral of order α of a function $e^s p(s)$, where $p(s)$ satisfies the conditions of Hausdorff transformed to the interval $(0, \infty)$. The conditions imposed on $p_1(s)$ by this requirement are of course well known. If α is an integer the corresponding conditions on $q_1(u)$ are easily obtained, viz.,

A necessary and sufficient condition that $[H, q_1(u)] \supset (C, n)$, $n \geqq 1$, *is given by:* (i) $q_1(u)$ *is an n-fold integral;* (ii) *the functions* $q_1(u)$, $uq_1'(u)$, $u^2q_1''(u)$, ..., $u^nq_1^{(n)}(u)$ *are of bounded variation in* (0, 1) *and tend to zero as* $u \longrightarrow 0$; (iii) *all these functions tend to one as* $u \longrightarrow 1$, *with the possible exception of* $u^n q_1^{(n)}(u)$.

A comparison of the criteria for $H_1 \supset (E, a)$ and for $H_1 \supset (C, \alpha)$ leads to the following observation. *If a method* $[H, q(u)]$ *includes both* (E, a) *and* (C, α) *then it includes their product as well.* This brings out quite clearly the lack of affinity between the two methods.

4. In our investigations we have been led to an algebraic paraphrase of the relative inclusion problem of Hausdorff which, perhaps, is not devoid of interest. Consider the set $\mathfrak{H}$ of all elements of the form $a\mathfrak{H}$ where a is any complex number and $\mathfrak{H}$ any Hausdorff matrix. It is easily shown that $\mathfrak{H}$ is a commutative domain of integrity. The unit matrix, which corresponds to convergence in the theory of limits, plays the rôle of unit element. There are in addition infinitely many units, i.e., elements whose reciprocals also belong to the set. These elements correspond to methods of summation equivalent to, but not coincident with, convergence. *The relative inclusion problem for Hausdorff matrices is equivalent to the factorization problem in* $\mathfrak{H}$.

The existence of the logarithmico-exponential scale of Hausdorff implies, among other things, the existence of an infinite sequence of matrices in $\mathfrak{H}$, $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3, \ldots, \mathfrak{H}_n, \ldots$ such that each element in the sequence is divisible by an arbitrary integral power of any one of the subsequent elements. The existence of one such sequence of course implies the existence of infinitely many. Thus we have a non-Archimedean system with respect to multiplication. Hence there exists a subdomain of $\mathfrak{H}$ in which no factorization of the elements into prime factors is possible. On the other hand it is easy to show that the moment functions

$$\mu(z) = \frac{\bar{a}}{a} \frac{a - z}{\bar{a} + z}, \quad \Re(a) > 0,$$

define matrices which are prime elements. The existence of further prime elements has to be left open for the present.

[1] F. Hausdorff, *Math. Z.*, **9**, 74–109, 280–299 (1921).
[2] R. Schmidt, *Math. Z.*, **22**, 89–152, 122–126 (1925).
[3] Loc. cit., §5–9 of the first paper.
[4] Among the various fairly recent proofs we notice several by Hausdorff. Cf. loc. cit and *Math. Z.*, **31**, 186–196 (1929).
[5] N. E. Nörlund, *Acta Math.*, **37**, 327–387, 366–368 (1914).

FINITE OVA

BY E. T. BELL

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated April 10, 1933

1. The algebraic varieties of most frequent occurrence in arithmetic are the ray, ring, module, group, semigroup and field. Rings are also called algebras and fields division algebras. Taking unique decomposition as the characteristic feature distinguishing arithmetic from algebra and analysis, we find varieties more rudimentary than any of the above as the abstract structures. In a previous paper* one type of these new varieties was called an ovum. Here we split the former ovum into yet simpler varieties, the entire class of which are now called ova. The several species of ova are defined by their postulate systems, stated presently. A few of the more important may be designated by specific mention of their characteristic properties; thus we have commutative ova, associative ova with a unity, etc.

2. The postulates for ova fall into four sections, concerning equality, composition, degree and unity, respectively. Not all the postulates in a section, except those concerning equality, appear in the definition of a particular ovum. The postulates for equality, together with one or more of those for composition, precisely one of those for degree, and either none, one or two of those for unity, define the class of ova. There are thus 72 species of ova (instances of all exist).

2.1 *Postulates for equality, E_0–E_4.*

E_0. There is a class K for which a binary relation, called equality and written $=$, is significant.

Elements of K will be denoted by small Latin letters. To indicate that K contains $x, y, \ldots$, we write $K \mid x, y, \ldots$.

E_1. If $K \mid x, y$, then $x = y$ or $x \neq y$, where $\neq$ is the negation of $=$.

E_2. If $K \mid x$, then $x = x$.

E_3. If $K \mid x, y, z$, and $x = y$, $y = z$, then $x = z$.

E_4. If $K \mid x, y$, and $x = y$, then $y = x$. The set E_0, E_1, E_2, E_3, E_4 will be denoted by E.

2.2 *Postulates for composition,* C_1-C_4.

C_1. If $K \mid x, y$, then there exists $C(x, y)$ such that $K \mid C(x, y)$ and $C(x, y)$ is unique.

We write $C(x, y) \equiv xy$, and call it the C-composition, or simply composition, of x, y, in this order.

C_2. If $K \mid a, b, x, y$ and $a = b, x = y$, then $ax = by$.

C_3. If $K \mid x, y, z$, then $C(C(x, y), z) = C(x, C(y, z))$. This may be written $((xy)z) = (x(yz))$. We write $((xy)z) \equiv xyz$.

C_4. If $K \mid x, y$, then $xy = yx$.

It is interesting to note that some modern sets of postulates for multiplication in rings, fields, etc., overlook C_2, whereas Euclid's Elements carefully states it. The postulate is necessary.

2.3 *Postulates for degree,* D_1-D_3.

If $K \mid x, y$ and $x \neq y$, x, y are said to be distinct.

D_1. K contains only a finite number of distinct elements.

D_2. K contains only a denumerable infinity of distinct elements.

D_3. K contains a non-denumerable infinity of distinct elements.

The number of distinct elements if D_1 is postulated is called the degree of K; in D_2, K is said to be denumerable, in D_3, non-denumerable. These definitions pass over to finite ova, denumerable and non-denumerable ova.

2.4 *Postulates for unity,* U_1, U_2.

U_1. If $K \mid x$, then $K \mid i$ and $ix = x$.

U_2. If $K \mid x$ and $K \mid i, j$ imply $ix = x, jx = j$, then $i = j$.

It is to be noted in the definitions of ova following that the omission of a particular postulate is merely an omission and not a contradiction of the postulate. Thus an associative ovum includes as subvarieties both commutative and non-commutative ova.

3. An *ovum* is defined by E, C_1, C_2; an *associative ovum* is defined by E, C_1, C_2, C_3; an *associative commutative ovum* is defined by E, C_1, C_2, C_3, C_4. These are the most important species in arithmetic, where the ova are either finite or denumerable, and where a unity as in U_1 is either postulated or not postulated, with the like for U_2 if U_1 is postulated.

4. If between the elements a_1, a_2, ... of one finite or denumerable ovum and those, a_1', a_2', ... of another, there is a correspondence such that, if a_i, a_i' ($i = 1, 2, \ldots$) are corresponding elements, and $a_i'\ a_j' = a_k'$ holds whenever $a_ia_j = a_k$ does, the ova are said to be simply isomorphic. This is simple isomorphism as for groups, rings, etc.; multiple isomorphism is defined in the usual way. Simply isomorphic ova are not distinguished in a census of finite ova of given degree. Ova of the same degree which are not simply isomorphic are said to be distinct (of that degree), with an obvious extension of this definition to denumerable ova.

The distinct associative, or associative commutative, finite ova of degree n can be determined non-tentatively in a finite number of steps from the

table of composition for degree n, which is analogous to the multiplication table for a group of order n. Let the elements be $a_1, \ldots, a_n$. At the intersection of row i, column j of the table is placed that one of $a_1, \ldots, a_n$ which is the composition of $a_i a_j$. For simplicity, $a_1, \ldots, a_n$ may be denoted by their suffixes and only the $n \times n$ matrix which is the body of the table need be given. There are not more than n^{n^2} distinct associative ova, or $n^{n(n+1)/2}$ distinct associative, commutative ova, of degree n. Both limits are much too high.

By straightforward methods all the associative commutative ova of a given degree can be written down from the possible symmetrical skeleton composition tables (symmetry about the principal diagonal) in which the principal diagonal is left blank. These skeletons may be examined for isomorphism before the diagonals are filled in; the surviving tables are then tested for associativity. This laborious process can be shortened by noting the subova (analogous to subgroups) in a given ovum.

The object of enumerating finite ova is to provide data for the study of abstract finite arithmetic.

The absence of a postulate for cancellation leads to many striking and unexpected properties. Although I personally have not been concerned with the arithmetic of commutative associative ova (finite or denumerable), I should like to mention that interesting, and in a sense final, results have been obtained in this direction by Mr. A. H. Clifford, of this Institute.

* E. T. Bell, "Unique Decomposition," *Am. Math. Month.*, **37**, 400–418 (1930).